TOPOGRAPHICAL OPTIMIZATION OF STRUCTURES FOR USE IN MUSICAL INSTRUMENTS AND OTHER APPLICATIONS

by

WILLIAM BRANDON KIRKLAND

LEE MORADI, COMMITTEE CHAIR
DAVID LITTLEFIELD, MECHANICAL ENGINEERING CHAIR
IAN HOSCH, PROFESSOR OF CIVIL ENGINEERING

A THESIS

Submitted to the graduate faculty of The University of Alabama at Birmingham,
In partial fulfillment of the requirements for the degree of
Master of Mechanical Engineering

BIRMINGHAM, ALABAMA

2014
TOPOGRAPHICAL OPTIMIZATION OF STRUCTURES FOR USE IN MUSICAL INSTRUMENTS AND OTHER APPLICATIONS

WILLIAM BRANDON KIRKLAND

MASTERS OF MECHANICAL ENGINEERING

ABSTRACT

Mallet percussion instruments such as the xylophone, marimba, and vibraphone have been produced and tuned since their inception by arduously grinding the keys to achieve harmonic ratios between their 1\textsuperscript{st}, 2\textsuperscript{nd}, and 3\textsuperscript{rd} transverse modes. In consideration of this, it would be preferable to have defined mathematical models such that the keys of these instruments can be produced quickly and reliably. Additionally, physical modeling of these keys or beams provides a useful application of non-uniform beam vibrations as studied by Euler-Bernoulli and Timoshenko beam theories.

This thesis work presents a literature review of previous studies regarding mallet percussion instrument design and optimization of non-uniform keys. The progression of previous research from strictly mathematical approaches to finite element methods is shown, ultimately arriving at the most current optimization techniques used by other authors. However, previous research varies slightly in the relative degree of accuracy to which a non-uniform beam can be modeled. Typically, accuracies are shown in literature as 1\% to 2\% error. While this seems attractive, musical tolerances require 0.25\% error and beams are otherwise unsuitable.

This research seeks to build on and add to the previous field research by optimizing beam topology and machining keys within tolerances that no further tuning is required. The optimization methods relied on finite element analysis and used harmonic modal frequencies as constraints rather than arguments of an error function to be
optimized. Instead, the beam mass was minimized while the modal frequency constraints were required to be satisfied within 0.25% tolerance.

The final optimized and machined keys of an A4 vibraphone were shown to be accurate within the required musical tolerances, with strong resonance at the designed frequencies. The findings solidify a systematic method for designing musical structures for accuracy and repeatability upon manufacture.
DEDICATION

For my wife,

Lauren Kirkland
ACKNOWLEDGMENTS

As this thesis is obviously inspired by musical instruments, I owe tremendous gratitude to all the instructors and teachers who helped me learn and perform in percussive arts.

I owe a world of gratitude to my mentor and advisor Dr. Lee Moradi of the UAB Center for Biophysical Sciences and Engineering, for his never ending support and enthusiasm; Dr. Littlefield, the Chair of the Department of Mechanical engineering at UAB for his encouragement; and Dr. Hosch, Professor of Civil Engineering at UAB for his support.

I have had tremendous support from my wife Lauren who has enabled me to pursue my educational goals through her understanding and encouragement. I also thank my family and friends for their support. But above all, I owe everything to God.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>iii</td>
</tr>
<tr>
<td>DEDICATION</td>
<td>v</td>
</tr>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>vi</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>x</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>xi</td>
</tr>
<tr>
<td>LIST OF ABBREVIATIONS</td>
<td>xiii</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>Definition of the Vibraphone</td>
<td>1</td>
</tr>
<tr>
<td>Traditional Methods of Hand Tuning</td>
<td>4</td>
</tr>
<tr>
<td>LITERATURE REVIEW</td>
<td>6</td>
</tr>
<tr>
<td>Timoshenko vibrations of non-uniform beams</td>
<td>6</td>
</tr>
<tr>
<td>Brief Background in Musical Intonation</td>
<td>8</td>
</tr>
<tr>
<td>Summary of Previous Research in Vibraphone Specific Analysis</td>
<td>11</td>
</tr>
<tr>
<td><em>Orduna-Bustamante (1991)</em></td>
<td>11</td>
</tr>
<tr>
<td><em>Bork (1995)</em></td>
<td>13</td>
</tr>
<tr>
<td><em>Bretos, Santamaria, and Moral (1997)</em></td>
<td>16</td>
</tr>
</tbody>
</table>
COMPUTER ASSISTED DESIGN AND OPTIMIZATION OF BEAMS FOR USE IN MUSICAL VIBRAPHONE INSTRUMENTS

Abstract ................................................................. 40

Research Design ......................................................... 41

Topology Definition ..................................................... 42

Meshing ......................................................................... 45

Frequency Definition and Optimization .................................. 45

Verification of Computer Optimized Geometry ............................ 46

Results and Analysis ....................................................... 47

Computational Results ..................................................... 47

Experimental Measurement ................................................ 51

Discussion ...................................................................... 54

FINAL CONCLUDING REMARKS ......................................... 57
SYSTEMS AND METHODS FOR DESIGNING AND FABRICATING MUSICAL INSTRUMENTS ............................................................. 60

REFERENCES .................................................................................................................................................. 84
## LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table I</td>
<td>44</td>
</tr>
<tr>
<td>Table II</td>
<td>52</td>
</tr>
</tbody>
</table>
**LIST OF FIGURES**

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Figure 1 – Three primary instruments of the mallet percussion family. From left to right: xylophone, marimba, and vibraphone.</td>
<td>2</td>
</tr>
<tr>
<td>Figure 2 – 1\textsuperscript{st}, 2\textsuperscript{nd}, and 3\textsuperscript{rd} transverse modes shown from top to bottom.</td>
<td>3</td>
</tr>
<tr>
<td>Figure 3 - Opposite side of a vibraphone key showing tuning curve.</td>
<td>5</td>
</tr>
<tr>
<td>Figure 4 – Isoharmonic tuning curves derived from Bernoulli-Euler beam theory. Circles indicate intersection of isoharmonic contours for first and second overtones.</td>
<td>12</td>
</tr>
<tr>
<td>Figure 5 – Isoharmonic tuning curves derived from Timoshenko beam theory. Circles indicate intersection of isoharmonic contours for first and second overtones.</td>
<td>12</td>
</tr>
<tr>
<td>Figure 6 – Local minima indicate locations of most effective tuning for a given mode.</td>
<td>14</td>
</tr>
<tr>
<td>Figure 7</td>
<td>15</td>
</tr>
<tr>
<td>Figure 8 – Meshes for the three cases studied by Bretos et al. (a) uniform cross section, (b) parabolic undercut, (c) rectangular undercut.</td>
<td>17</td>
</tr>
<tr>
<td>Figure 9 – Frequency ratios of a 5 octave Malletech marimba. (a) transverse mode vibrations (b) torsional mode vibrations.</td>
<td>30</td>
</tr>
<tr>
<td>Figure 10 – Three functions considered for optimization by Petrolito and Legge.</td>
<td>32</td>
</tr>
<tr>
<td>Figure 11 – Optimization of a 1:3:5:8 vibraphone bar using trigonometric functions.</td>
<td>33</td>
</tr>
<tr>
<td>Figure 12 – Optimization of a 1:3:5:8 resonator with Chebyshev polynomials.</td>
<td>35</td>
</tr>
<tr>
<td>Figure 13 – Symmetric ( \frac{1}{2} ) thickness and 1/50 length cuts studied for effect on modal frequency.</td>
<td>37</td>
</tr>
</tbody>
</table>
Figure 14 – Influence of symmetric cuts on transverse modal frequency as predicted by FEA. ................................................................. 38

Figure 15 – The resulting length of two methods graphed. Note, only natural keys are pictured. ........................................................................................................ 48

Figure 16 – Node 1 represents those closest to the musician. Node 2 represents those closer to the audience. ........................................................................................................ 49

Figure 17 – Drawing profiles for each instrument optimization. (Top) Length variable and optimized to reduce mass. (Bottom) Length constrained and only the spline optimized........................................................................................................ 50

Figure 18 – Final machined key geometries. Both keys pictured were machined according to optimized parameters for an A4 vibraphone key. .................................................. 51

Figure 19 – Spectral frequency plot of key 1. ........................................................................................................ 53

Figure 20 – Spectral frequency plot of bar 2. ........................................................................................................ 53
LIST OF ABBREVIATIONS

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>CAD</td>
<td>computer assisted design</td>
</tr>
<tr>
<td>CNC</td>
<td>computer numeric code</td>
</tr>
<tr>
<td>FEA</td>
<td>finite element analysis</td>
</tr>
<tr>
<td>FEM</td>
<td>finite element method</td>
</tr>
<tr>
<td>PACS</td>
<td>Physics and Astronomy Classification Scheme</td>
</tr>
</tbody>
</table>
INTRODUCTION

The topic of this thesis presents the application of mechanical engineering practice to the design and manufacture of musical instruments. In particular this work and literature review focuses on the vibraphone. However, the principles discussed herein may be applied to a wide variety of application, music and beyond. Here the essential concept and basic physics of vibraphones will be introduced, with the detailed discussion and analysis to follow in later chapters.

Definition of the Vibraphone

At the very simplest level, percussion instruments represent a large subclass of musical instruments. Percussion instruments are widely regarded as any instrument which produces sound by striking it (impulse excitation). Obviously this includes drums as well as xylophones and similar instruments. Due to the broad definition of percussion instruments, some consider instruments such as the piano to be a percussion instrument. This is because the strings are struck by hammers, actuated upon the press of a key. Clearly a more precise term is needed to further narrow the specific instruments of study.

The term “Idiophones” further narrows the previous field by excluding those instruments whose sound is produced by strings or membranes. This new distinction removes drums and the previously mentioned piano from consideration. What remains
are those instruments whose sound originates from the body of the instrument. This includes xylophones, cymbals, and similar instruments.

Finally refining the classification of instruments a third time yields the class of this paper’s consideration: “mallet percussion”. This classification limits itself to instruments which are played by striking (percussion) bars (idiophones) with handheld mallet actuators. The number of instruments within this category is relatively minimal, including xylophones, vibraphones, marimbas shown in Figure 1 and other similar instruments. Each of these three instruments is played with largely identical methods. However distinctions between each instrument need to be made.

![Figure 1](image)

Figure 1 – Three primary instruments of the mallet percussion family. From left to right: xylophone, marimba, and vibraphone.

Xylophones are likely the most familiar due to many children’s toys being fashioned in their likeness. Professional xylophones likely to be found in an orchestra are characterized by their ‘short’, ‘brittle’ tones. Xylophone keys are made of varying types of wood or plastic and have significant internal material damping, such that the acoustic sustain is minimal. Finally, xylophones can be identified aurally by their timbre, described as follows. Each key of a xylophone has a unique fundamental frequency. This fundamental frequency defines the note name of the bar (e.g. A, B, C#). A xylophone of
good timbre must also have clearly defined harmonics of the fundamental frequency. These harmonics must be in the ratio of 1:3:6. Explaining further, the fundamental mode of vibration is a transverse mode. The second transverse mode must have a frequency of 3 times the first. The third transverse mode must have a frequency of 6 times the first. For clarification, Figure 2 shows the mode shapes for each of these transverse vibrations.

Figure 2 – 1st, 2nd, and 3rd transverse modes shown from top to bottom.

Similar to xylophones, marimbas are constructed from rosewood, synthetic wood, or exotic African woods such as padouk. The sound of a marimba is characterized as ‘warm’. The material still naturally damps itself but the sustain of the note is much longer than the xylophone. Marimba timbre is typically found in the ratios of 1:4:10. Again as in the previous example, this stipulation requires the second transverse mode to have a
frequency of 4 times the first, and the third transverse must have a frequency of 10 times the first.

Finally, vibraphones are constructed with aluminum bars. The most striking difference caused by this change is the length of sustain in vibraphones due to the inherently low internal damping of the material. Similar to the marimba, vibraphones have a timbre defined by the ratio 1:4:10. While the engineering analysis and optimization techniques of this work are applicable to all three instruments, the vibraphone is uniquely well suited to experimentation. Of the three instruments and their materials, the vibraphone (with aluminum keys) is the only one with isotropic material properties and reliably consistent mechanical properties.

The papers referenced in the Literature Review of this thesis consider studies on the bars of all three instruments. Again, the primary differences between the bars of the three instruments are only material properties and harmonic tuning ratios. As such, the governing physics and theorems are applicable to all three instruments.

Traditional Methods of Hand Tuning

The following traditional tuning methods apply equally well to vibraphones, marimbas, and xylophones. Keys of each instrument are tuned individually through a labor intensive process. Parameters such as width, length, and thickness are defined by manufactures and are not allowed to vary from a predetermined value. Thus, instrument technicians tune each key by grinding a curvature on the underside, roughly symmetric about the center. Figure 3 displays an example set of vibraphone keys in which the tuning
The action of grinding each key reduces the local mass at a given location. This reduction of material also reduces the mechanical stiffness at that point where mass was removed. Therefore, one can quickly surmise reducing the thickness in the center of the key will lower the fundamental frequency of the key.

Tuning the harmonics to achieve an appropriate timbre requires more strategic placement of mass removal. Grinding outside the midpoint, moving towards the ends of the bar most effectively tunes the second transverse mode, similarly moving out even further from the center tunes the third transverse mode. However removing a given mass affects all three modes of vibration to varying degrees dependent on only the location of the grinding.
LITERATURE REVIEW

Timoshenko vibrations of non-uniform beams

Any research into the vibrations of non-uniform beams will immediately return results based on Timoshenko’s work. Therefore it makes the most obvious place to start formulations within this thesis. Timoshenko begins his derivations with his previously derived relationship for beam vibration.

\[
\frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 y}{\partial x^2} \right) + \rho A \frac{\partial^2 y}{\partial t^2} = 0
\]

Equation 1

From this expression, Timoshenko presents a relation for angular velocity \( \omega \).

\[
\omega^2 = \frac{E}{\rho} \frac{\int_0^l l \left( \frac{d^2 X}{dx^2} \right)^2 dx}{\int_0^l A X^2 dx}
\]

Equation 2

Where \( X \) is defined as the mode shape and is assumed to be a series of the following form.

\[
X = a_1 \phi_1(x) + a_2 \phi_2(x) + a_3 \phi_3(x) + \ldots
\]

Equation 3
Every function $\phi_n(x)$ must satisfy all conditions at the end of the beam. For example, the vibraphone beams modeled here in this paper are all treated as free-free beams. Thus, a simple trigonometric sine function would be enough to satisfy the end conditions: inflection at $x = 0$ and $x = l$ equals 0. (Timoshenko, Young, & Weaver, Jr., 1974)

Using these formulations, Timoshenko in his work *Vibrational Problems in Engineering* presents several example calculations. However each case considers only beams which are symmetric about their length axis.
Brief Background in Musical Intonation

The subject of this thesis revolves around optimal tuning according to harmonic ratios for applications in music. Therefore, it’s worthwhile to devote some brief discussion to the basics of musical intonation.

The most basic building block of any musical scale is the octave. If two frequencies are equal, they are said to be in tune. If two frequencies are played simultaneously such that \( f_1 = 2f_2 \) then an octave relationship is said to exist between the two notes. This relationship is noted to be acoustically pleasing and pleasant from a human psychological perspective. More musical octaves may be added as in the following series.

\[
1: 2: 4: 8: 16
\]

Referring to the series above, if five strings were tuned such that each string was two times the frequency of the last, an octave exists between each string. Mathematically, the series may be called a log base two series. However octaves alone are not enough for a musician to evoke an array of human emotions. For example, in a horror film the composer would not want “pleasant” octaves. However with two frequencies separated relatively slightly the composer can achieve ‘dissonance’ which sounds unpleasant. Thus, the musical scale is not defined by the octave alone but instead is defined by how octave ratios are broken up and subdivided.

In the popular music of today twelve notes exist within the octave. Bearing in mind the already established logarithmic nature of the octave series, twelve notes may be spaced equally as follows.
This final expression for $f$ represents the interval relationship between adjacent musical notes regardless of octave (Olson, 1967).

As a matter of ease and convenience, musicians measure frequency intervals in units of cents. Simply put, a value of 100 cents is assigned to the ratio $\sqrt[12]{2}$, and 1,200 cents exists within the whole octave. A formula for cents can be written as follows.

$$cents = 1200 \log_2(f_{ratio})$$

Where $f_{ratio}$ is written to clarify a ratio between two frequencies $f_{ratio} = \frac{f_1}{f_2}$. The exact frequency sensitivity of human hearing varies from person to person and with age among other factors. However standard musical tolerance is considered to be within plus or minus five cents. If one were to solve for $f_{ratio}$ using 5 cents, it can be shown the resulting ratio corresponds to an error of 0.289 %.

To this point, the definition of an octave as simply a 1:2 ratio and further subdivisions of that ratio has been established. Therefore, a musical scale could be made up from any starting frequency as long as the precise ratios are maintained. Today, most orchestras tune to $A_4 = 440$ Hz. The letter defines the note name and the subscript defines the octave. Therefore $A_5 = 880$ Hz. To give another example to clarify note naming, frequency ratios could be written again as follows.

$$1, f^1, f^2, f^3, f^4 \ldots f^{12}$$

$$where \quad f^{12} = 2$$

$$or \quad f = \sqrt[12]{2}$$
In the example above, # is used in musical notation to indicate sharp keys. These keys are the black keys seen on a piano. Additionally, the numeric subscript which defines the octave is seen to increment at C rather than A. This is typical of music terminology. The pattern above could be shown to beyond A₅, and it would repeated in a similar fashion. Though A₄ = 440 Hz is the established norm, it is truly a matter of personal preference and some composers have chosen alternate values throughout history. Important to this thesis, although an orchestra may tune to A₄ = 440 Hz, most professional percussion instruments including vibraphones are tuned to A₄ = 442 Hz. Once again, this is simply a matter of preference as most composers feel the slightly ‘brighter’ pitch ‘blends’ well with the rest of the orchestra. Indeed, music forms a very real link between physics, engineering, mathematics, psychology, and human anatomy.

Finally, the ratios defined above are named as the Musical Scale of Equal Temperament. Other musical ratios can be defined such as the Scale of Just Intonation however its definition falls outside the scope of this thesis. From a historical perspective, the Scale of Equal Temperament was popularized during the Baroque period and by composers such as J.S. Bach.
Summary of Previous Research in Vibraphone Specific Analysis

Orduna-Bustamante (1991)

Orduna-Bustamante was among the first researchers to investigate optimal tuning topology for applications to mallet percussion instruments, specifically the marimba. Orduna-Bustamante points out a free-free beam with constant cross section will result in natural frequency ratios of 1:2.76:5.4. These ratios relate to the first, second, and third transverse mode shapes respectively. Orduna-Bustamante sought to develop tuning curves for marimba bars which would result in harmonic ratios of 1:4:9.88. From a musical viewpoint, the 9.88 ratio was preferred in a study performed by Bork and Meyer and referenced by Orduna-Bustamante.

Relying on the Bernoulli-Euler and Timoshenko mathematical models for beam vibration, Orduna-Bustamante considered the general case of beam thickness, \( t(x) \) defined with a parabolic undercut.

\[
t(x) = T_c + (1 - T_c) \left( \frac{x}{X_c} \right)^2 \quad \text{if } |x| < X_c
\]

\[
t(x) = 1 \quad \text{if } |x| > X_c
\]

Equation 4

Where \( X_s \) and \( T_s \) are dimensionless parameters controlling the width of parabolic cut and depth of the cut at the center.

From this geometric definition, Orduna-Bustamante derived isoharmonic contours from the Bernoulli-Euler and Timoshenko models. These isoharmonic tuning curves are repeated from Orduna-Bustamante’s original work in Figure 4 and Figure 5.
Figure 4 – Isoharmonic tuning curves derived from Bernoulli-Euler beam theory. Circles indicate intersection of isoharmonic contours for first and second overtones.

Figure 5 – Isoharmonic tuning curves derived from Timoshenko beam theory. Circles indicate intersection of isoharmonic contours for first and second overtones.
Given a uniform beam with a known fundamental frequency, the tuning curves are utilized by finding the number of semitones (100 cents) to the target fundamental frequency. Second, the required harmonic ratio must be known.

Orduna-Bustamante noted in reference to the above figures the problem cannot be solved in all cases. However, the curves were proven experimentally to be accurate to within 7 cents error. (Bustamante, 1991)

With respect to the vibraphone topology optimization presented herein, parabolic undercuts following Orduna-Bustamante’s research were initially investigated as a preference to cubic-spline curves owing in large part to the relative simplicity with fewer variables to optimize. However to achieve 1:4:10 harmonic ratios in the low register of the instrument required beams of unreasonable length if parabolic undercuts were to be employed. However, it is worth noting the spline defined topology converges to a roughly parabolic shape in the mid-range frequency keys of the instrument.

*Bork (1995)*

Bork in 1995 produced one of the early papers regarding the challenges of modern xylophone, marimba, and vibraphone tuning. His investigations were published in *Practical Tuning of Xylophone Bars and Resonators*. It should be noted, his findings apply equally well to all mallet percussion instruments. Typical mode shapes have already been pictured in this thesis in Figure 2. Among many of the covered topics, Bork developed a computational model for tuning which has been reproduced in Figure 6. In essence, Bork showed that to tune a given mode, mass must be removed from the location of highest displacement in its mode shape.
Figure 6 – Local minima indicate locations of most effective tuning for a given mode. Again referring to the Figure 6, if the third harmonic were to be tuned at a given ratio of 10:1 then mass would need to be removed at the minima of the plotted line on the graph. General tuning procedure requires one to tune the ratios first and then tune the fundamental by removing mass as shown by the minima location of the 1.0 curve. Later, Laukkanen and Worland would expand on this concept by generating locations where one mode can be tuned without affecting the relative ratio of other modes.

Although not typically covered by authors in the field, Bork investigated optimal tuning and cross-section shape of resonators as well. As already seen, the cross-section of typical instrument resonators is round. However Bork shows in Figure 7 the sound field
is roughly elliptical. Therefore, better power transfer can be achieved by producing resonators to match such a shape.

Figure 7 – Constant pressure curves show the sound field radiated to be roughly elliptical.

After study, elliptical resonators were found by Bork to result in decibel increases of roughly 6 dB. Of course, no effective transfer of power comes without cost and in this process...
case an increase in volume results in significant decrease in sustain time. (Bork, Practical Tuning of Xylophone Bars and Resonators, 1995)

This tradeoff may or may not be attractive to musicians at large. For instance, xylophones are generally intended to have very short sustain times but vibraphones are well known for their long and almost ‘bell like’ sustain.

_Bretos, Santamaria, and Moral (1997)_

Bretos et al. were among the first to introduce FEA modeling to mallet percussion instruments, in this case using ABAQUS software. Their work considered three cases (uniform cross section, parabolic undercut, and rectangular undercut) modeled with block elements reproduced in Figure 8. The parabolic undercut was chosen to result in a tuning ratio of 1:4 and was developed from Orduna-Bustamante’s work. The second overtone, third transverse, was not designed to a harmonic ratio. The rectangular undercut was designed to yield the harmonic ratio 1:3. This ratio would be appropriate for a xylophone.
Figure 8 – Meshes for the three cases studied by Bretos et al. (a) uniform cross section, (b) parabolic undercut, (c) rectangular undercut.

Experimental results shows the fundamental frequency of the parabolic undercut was 1.2% lower than the target frequency of $E_4$. Similarly the rectangular undercut resulted in deviations less than 1.1% from the target value. (Bretos, Santamaria, & Moral, 1997)
The findings of Bretos et al. establish FEA modeling as a viable method for analyzing mallet percussion bars. However, the deviations of 1.2% and 1.1% for the bars modeled and tested correspond to deviations of -20 cents and 17 cents respectively. For reference, standard musical tolerances are generally within ±5 cents.

*Chaigne, Doutaut, Matignon (1998)*

Chaigne introduced two important papers in 1997 and 1998. The first part, published earlier, presented a highly complete numerical model of transient mallet-bar impact in the time domain. This model was based on Bernoulli-Euler method and included viscous beam dampening, as well as mallet mass and impact velocity. The end result was numerically predicted waveforms with striking similarity to waveforms measured experimentally from xylophone instruments. The governing equations assumed by Chaigne et al from Bernoulli-Euler theory are as follows.

$$M(x,t) = -EI(x)\left(1 + \eta \frac{\partial}{\partial t}\right) \frac{\partial^2 w}{\partial x^2} (x,t)$$

$$\frac{\partial^2 w}{\partial x^2} (x,t) = \frac{1}{\rho S(x)} \frac{\partial^2 M}{\partial x^2} (x,t) - \frac{X}{M_B} \frac{\partial w}{\partial t} (x,t) - \lambda_b \frac{\partial w}{\partial t} (x,t) - f(x, x_0, t)$$

*Equation 5*

With the boundary conditions of a free-free beam, the second and third derivative of displacement $w$ about $x$ can be reduced to zero.

$$\frac{\partial^2 w}{\partial x^2} (x,t) = \frac{\partial^3 w}{\partial x^3} (x,t) = 0 \bigg|_{x=0,L}$$

*Equation 6*
In the equations above, a few variables are particularly important for the time domain modeling of the xylophone key. \( \eta \) represents viscoelastic losses which Chaigne defined by a relation between stress and strain.

\[
\sigma(x, t) = E \left( \epsilon(x, t) + \eta \frac{\partial \epsilon}{\partial t}(x, t) \right)
\]

Equation 7

Interestingly, the variable \( \lambda_b \) designates fluid damping. However Chaigne notes it provides a good representation for losses in wooden bars. Finally, mallet-bar interaction forces are governed by the final term \( f(x, x_0, t) \).

\[
f(x, x_0, t) = f(t)g(x, x_0)
\]

\[
f(t) = \frac{F(t)}{\rho S(x) \int_{x_0-\delta x}^{x_0+\delta x} g(x, x_0) dx}
\]

Equation 8

\( F(t) \) represents the impact force of the mallet onto the bar. The function \( g(x, x_0) \) indicates the distribution of force over the width of impact. As Chaigne points out, this function may be defined as a simple triangular or cosine function. Variations on this function were not observed to cause significant changes to final results. Finally, the impact force and mallet motion is defined by classical Newton physics.

\[
F(t) = -m_e \frac{d^2 \zeta(t)}{dt^2}
\]
Chaigne et al. continued their numeric formulations on two methods. First, for beams of uniform cross-section a relatively simple explicit finite difference method was used. However, for a non-uniform beam, fourth order partial derivatives and the viscous damping forces required an implicit method. Chaigne et al. presented the uniform and non-uniform bar calculations in their work however those formations will not be reproduced here in full. However it is necessary to show the governing equation for a beam with uniform thickness. This equation is the result of imposing constant thickness on the cross section $S$ in Equation 5.

$$
\frac{d\zeta(0)}{dt} = V_0
$$

Equation 9

Approximating the equation above was done by explicit finite differences which require the bar to be divided into $N$ steps. In general for the implicit case, the maximum number of steps in which the bar can be divided is limited by an assumed sampling frequency, and accurate calculations of frequencies become increasingly intensive.

In the explicit case, the fourth order partial derivative is approximated by use of the theta method. For clarification, the theta method is sometimes referred to as the weighted method or theta schemes. Chaigne et al. showed for very large values of $N$ ($N = 552$),
frequencies can be calculated to within 0.3% error (Chaigne & Doutaut, 1997). This finding is striking for its accuracy. As a reminder, musical tolerance of 5 cents would require 0.289 % error at minimum. Chainge et al. continued further, showing their numeric model could be applied to any cross section such as the parabolic shapes previously studied by Orduna-Bustamante.

_Bork, Chaigne, Kosfelder, and Pillot (1999)_

Bork et al. further validated FEA methods applied to bar percussion instruments. Their work investigated modal analysis (performed experimentally and with Fast-Fourier Transforms) and FEA Modal modeling of a C3 (130 Hz) rosewood marimba bar. Their experimental data collection included accelerometer measurements as well as microphone measurements.

The FEA Methods employed by Bork et al. assumed geometry of constant thickness about its profile. This is despite variations of thickness by as much as 20% as found by the team. Non-uniform meshes were employed to reduce computation time with highest mesh density in the center of the beam span where deflection is greatest. The total number of elements employed for the analysis was 2880 cubic elements.

Initial FEA calculations were performed assuming isotropic material. This method resulted in accurate calculations of only the lowest modal vibrations and the team moved forward to model the rosewood material as a true composite with orthotropic properties. However, it’s worth noting that some variations in material properties such as Poisson’s ratio and shear modulus had negligible impact on the calculated eigenfrequencies.
Final comparisons between experimental and numeric results indicated accuracies of approximately 4%. One low frequency torsional mode was calculated with 0.6% accuracy. (Bork, Chaigne, Trebuchet, Kosfelder, & Pillot, 1999)

This early work by Bork et al. once again holds similarities to the work by Bretos as well as the work presented here in this thesis. It is the opinion of this author that the inaccuracies in Bork et al.’s findings can likely be attributed to two factors. First, experimental modal analysis was performed in part by placing accelerometers on the surface of the key. Accelerometers have been observed to cause significant changes in resonant frequency when applied to vibraphone keys. The C3 bar studied by Bork is very large in mass compared to most other keys of the instrument. Even so, the presence of accelerometers likely held some impact. Secondly, FEA analysis was performed with only 2880 cubic elements and required 10 hours to run as acknowledged by Bork. This is simply a limitation of the technology at the time. The work presented in this thesis required as many as 10,000 tetrahedral elements to achieve musical tolerances.

B. H. Suits (2001)

A mathematical description of vibrating beams with non-uniform cross-section has already been touched upon by the previous formulation presented from Timoshenko. Many of the authors already discussed adapted Timoshenko’s beam models to the applications at hand. B.H. Suits however provides the most in depth overview of Timoshenko beam applications to marimba and xylophone bars.

In general, Suits found Bernoulli-Euler beam theory to be far too imprecise when considering bars of the size required in marimbas and other musical applications.
However, Timoshenko theory can be accurately applied to non-uniform bars with the only restriction being the wavelength must be large compared to the width of the bar.

*Vibrations of uniform bars according to Timoshenko theorem.* For the moment, the general formulations for beams of uniform cross section are considered. Suits showed the forces acting on a given element can be reduced to the following shear $Q$ and moment $M$ forces if longitudinal and lateral vibrations are to be ignored.

$$M = EI\epsilon$$  

Equation 11

$$Q = kGS\gamma$$  

Equation 12

$\epsilon$ and $\gamma$ denote the angle of deformation resulting from the moment and shear forces respectively.

If $\gamma(x, t)$ is taken to be the vertical deformation of the beam, then for small displacements the following relation is given.

$$\frac{dy}{dx} = \psi + \gamma$$  

Equation 13

From these relations, Suits presents Timoshenko’s result for rotational and vertical motion.
\[ EI \frac{\partial^2 \psi}{\partial x^2} + KGS \left( \frac{\partial y}{\partial x} - \psi \right) - \rho l \frac{\partial^2 \psi}{\partial t^2} = 0 \]

Equation 14

\[ \rho S \frac{\partial^2 y}{\partial t^2} - KGS \left( \frac{\partial y}{\partial x} - \frac{\partial \psi}{\partial x} \right) = 0 \]

Equation 15

For which the general solution is presented as:

\[ y(x, t) = e^{i\omega t} Y(x) \quad \psi(x, t) = e^{i\omega t} \psi(x) \]
\[ Y(x) = C_1 \cosh b\alpha \xi + C_2 \sinh b\alpha \xi + C_3 \cos b\beta \xi + C_4 \sin b\beta \xi \]
\[ \psi(x) = C'_1 \sinh b\alpha \xi + C'_2 \cosh b\alpha \xi + C'_3 \sin b\beta \xi + C'_4 \cos b\beta \xi \]

Equation 16

Where \( \xi = \frac{x}{t} \). The other variables are defined as:

\[ b^2 = \frac{\rho S L^4}{EI} \omega^2 \quad \rho^2 = \frac{I}{SL^2} \quad s^2 = \frac{E}{kG} r^2 \]

\[ \alpha, \beta = \left\{ \left( r^2 - s^2 \right)^2 + \frac{4}{b^2} \right\}^{\frac{1}{2}} + \left( r^2 - s^2 \right)^{\frac{3}{2}} \]

Equation 17

Finally, Suits simply defines \( C'_j \) as proportional to \( C_j \). However Tong et al gives a more complete definition (Tong, Tabarrok, & Yeh, 1995).

\[ C'_1 = \frac{b(\alpha^2 + s^2)}{L\alpha} C_2 \]
Equation 18

\[ C'_{2} = \frac{b(\alpha^2 + s^2)}{L\alpha} C_{1} \]

\[ C'_{3} = \frac{b(\alpha^2 + s^2)}{L\alpha} C_{4} \]

\[ C'_{4} = \frac{b(\alpha^2 + s^2)}{L\alpha} C_{3} \]

Boundary conditions are simply described as a free-free or ‘floating’ condition, meaning the shear and moment forces must equal zero at \( x = 0 \) and \( x = L \).

\[ \frac{\partial \Psi}{\partial x} = 0 \]

\[ \frac{\partial Y}{\partial x} - \Psi = 0 \]

Equation 19

As referenced by B.H. Suits, the equation resulting from substitution of Equation 16 into Equation 19 results in four homogenous equations. Suits does not present the calculation, however it is shown here for completeness.

at \( x = 0 \)

\[ \frac{b^2 C_2 (\alpha^2 + s^2)}{L^2} + \frac{\beta b^2 C_4 (\alpha^2 + s^2)}{\alpha L^2} = 0 \]

\[ \frac{ab C_2}{L} + \frac{\beta b C_4}{L} - \frac{b C_1 (\alpha^2 + s^2)}{\alpha L} - \frac{b C_3 (\alpha^2 + s^2)}{\alpha L} = 0 \]
Finally, the four homogeneous equations of Equation 20 can be rewritten into a matrix format according to the coefficients $C_j$. Again, this matrix is not shown by Suits however is shown here for its importance to this thesis.

\[
\begin{pmatrix}
0 & \frac{b^2 \sinh(ab)}{L^2} & 0 & \frac{\beta b^2 \cos(\beta b)}{\alpha L^2} \\
0 & \frac{b \cosh(ab)}{L} & \frac{b^2 \sinh(ab)}{L} & \frac{\beta b^2 \cos(\beta b)}{\alpha L^2} \\
\frac{\alpha b \sinh(ab)}{L} & \frac{b \cosh(ab)}{L} & \frac{b^2 \cos(\beta b)}{\alpha L} & \frac{\beta b \sin(\beta b)}{\alpha L} \\
\frac{\alpha b \cosh(ab)}{L} & \frac{b \cosh(ab)}{L} & \frac{b^2 \cos(\beta b)}{\alpha L} & \frac{\beta b \sin(\beta b)}{\alpha L} \\
\end{pmatrix}
\]

where

\[
\sigma_1 = -\frac{b \sigma_2}{\alpha L} \\
\sigma_2 = \alpha^2 + s^2
\]
Taking the determinate of the matrix above finds the nontrivial solution. Suits reports the satisfying equation to be as follows.

\[
2 - 2 \cosh(b\alpha) \cos(b\beta) \\
+ \frac{b}{(1 - b^2r^2s^2)^{\frac{1}{2}}} \left[ \frac{b^2r^2(r^2 - s^2)^{\frac{1}{2}} + (3r^2 - s^2)}{2} \right] \sinh(b\alpha) \sin(b\beta) = 0
\]

Equation 21

Recalling that variable \( b \) can be considered as a function of frequency \( \omega \), the modal frequency can be determined quickly by using a computational root solver algorithm.

Once again, Equation 21 was developed for uniform bars. The proceeding sections will develop the equations further for application to non-uniform bars.

Extension of Timoshenko beam theory to beams of variable cross section. Some limitations are imposed in the process of extending the governing Timoshenko equations to beams of varying cross section. The beam is assumed to have only one degree of freedom. This disregards torsional and lateral vibrations resulting in some error. To apply the previous formulations, the vibraphone beam is thought to be broken into many cross sections about the lengths such that each section is that of a simple rectangle. Each section is assumed to have a set length or thickness, which approximates the original bar if they were to be stacked together. In this way, each section has the same general solution as presented by Suits in Equation 16.

Boundary conditions still include those defined above. Shear and Moment equal zero at the bar ends. However, new boundary conditions must be added to account for the
interface between sections \( \nu \) and \( \nu + 1 \). These boundary conditions preserve continuity of shear and moment stresses across the length of the beam.

\[
I_{\nu} \frac{\partial \Psi_{\nu}}{\partial x} = I_{\nu+1} \frac{\partial \Psi_{\nu+1}}{\partial x}
\]

\[
S_{\nu} \left( \frac{\partial Y_{\nu}}{\partial x} - \Psi_{\nu} \right) = S_{\nu+1} \left( \frac{\partial Y_{\nu+1}}{\partial x} - \Psi_{\nu} \right)
\]

Equation 22

Again, in the sections bounded by \( x = 0 \) or \( x = L \) the appropriate boundary conditions from Equation 19 should be used. Piecewise discretization of the non-uniform beam as defined here results in four \( C_j \) constants for each region \( \nu \). As Suits points out, this results in 4N total coefficients for N number of regions. The coefficients for a given region can be solved identically as before. Thus it quickly becomes apparent that any meaningful and accurate calculation requires a large number of coefficients. Such calculation almost certainly requires computational programming to be solved.

(Suits, 2001)

Tong, Tabarrok, and Yeh have already been mentioned above. Their work on *Vibration Analysis of Timoshenko Beams with Non-Homogeneity and Varying Cross-Section* was published several years prior to Suits, 1995. However it’s interesting to point out the work covers beams with varying density and material properties as well (Tong, Tabarrok, & Yeh, 1995). Although, such a calculation is generally unnecessary in the case of percussion beams for musical instruments, even in the case of composite materials such as rosewood marimba bars.
Rossing, Yoo, Morrison (2004)

Rossing et al. are included in this literature review for their consideration of percussion instruments as a whole, but particularly for their useful study of a Malletech marimba’s frequency ratios. Figure 9 (a) shows transverse bending modes demonstrating the strongly harmonic series for the low frequencies. The third transverse mode frequency is allowed more deviation in higher modes. However, it’s worth noting that many of those frequencies are well within the limits of human hearing if we accept the upper limit as 20000 Hz. Figure 9 (b) shows torsional modes and their frequency ratios. These frequencies are not as tightly controlled and appear to be tuned only in the low register of the instrument. Torsional mode tuning is necessitated by the wider and thinner bars of the bass section of the instrument. (Rossing, Yoo, & Morrison, 2004)
Figure 9 – Frequency ratios of a 5 octave Malletech marimba. (a) transverse mode vibrations (b) torsional mode vibrations.
The included frequency ratios provide significant insight into the production of percussion instruments in industry. Although the high frequency third transverse harmonics may be difficult to hear, they are none the less within the range of human hearing. The shape topology optimization procedure discussed later was successful in producing FEA models indicating all fundamental and 2 overtones of each key could be produced. However further study will be needed to determine the accuracy at higher regions of the instrument.

*Petrolito and Legge (2005)*

Designing musical structures using a constrained optimization approach

Petrolito and Legge sought to optimize the undercut of the marimba key using varying functions to define its geometry. Among the three functions investigated were piecewise (step), piecewise-linear, and sine. These three functions are graphically displayed in Figure 10. For clarification, the sine function was defined not unlike a Fourier series.

\[
h(x) = h_{max} - \frac{2a_0}{L_2} x - \sum_{N=1}^{N-1} a_N \sin \left( \frac{2n\pi x}{L_2} \right) \quad x \leq \frac{L_2}{2}
\]

Equation 23
Figure 10 – Three functions considered for optimization by Petrolito and Legge.

(Petrolito & Legge, 2005)

*Luis Henrique, Octavio Inacio, Jose Paulino, Jose Antunes* (2005)

Optimization of Vibratory and Acoustical Components of Percussion Instruments:

Theoretical and Experimental Results
Henrique et al. studied the problem at hand with methods that inspired the work presented here. Acknowledging the necessity to simplify the number of variables to optimized, trigonometric and Chebyshev functions were selected; however Bezier and spline curves were also recognized as a possible approach. Additionally, Henrique et al. recognized the bars could be optimized to achieve any set of harmonic ratios desired and optimized a few example designs as shown in Figure 11.

![Figure 11 – Optimization of a 1:3:5:8 vibraphone bar using trigonometric functions.](image)

Importantly, Henrique et al. expanded their scope to include resonators as found on marimbas and vibraphones. The motivation for this addition was addressed in one
significant flaw of standard close-ended resonator design. Such resonators are capable of resonance of only odd numbered harmonic ratios (e.g. 1:3:5). In the case of vibraphones and marimbas tuned with standard 1:4:10 ratios, only the fundamental frequency can be amplified. Almost identically to the optimization of bar shape, a variable cross sectional area was optimized over the length of a given resonator (Figure 12). Finally, Timoshenko beam model was employed as in the previous papers. For the acoustical component of resonators, one dimensional wave modeling was used.
Figure 12 – Optimization of a 1:3:5:8 resonator with Chebyshev polynomials. (Henrique, Inacio, Paulino, & Antunes, 2005)

Although the work presented here in this thesis was significantly inspired by the work of Henrique et al., a few key differences remain. First, Henrique found a limited number of elements (hundreds) were sufficient to accurately predict frequencies and eigenvalues (compared to an analytical equivalent). As will be shown in the work here, as many as 10,000 elements were deemed necessary to ensure satisfaction of musical
tolerances upon machining. Another minor but important difference, the work presented in this thesis includes node holes in the optimized final and machined design. For clarification, these node holes are through-holes where the key will rest upon strings once assembled into the final instrument. Because these through holes obviously pass through the nodes of the fundamental mode, the first transverse mode is not significantly affected. However these node holes have some impact on higher order modal vibrations. A final but critical difference, Henrique et al. performed their optimization by minimizing an error function, effectively minimizing the maximum error. However in this thesis, the total mass of the key was minimized and the target frequencies were held as design constraints.

Henrique et al.’s work on resonator optimization is exceptionally interesting. Obviously, by tuning a resonator to amplify three modes as opposed to one, an increase in instrument volume will result. However as already shown by Bork, this increase in power transfer will come at the price of reduced instrument sustain. Because the varying diameter resonators were tested with a speaker producing a continuous waveform, no measurements indicating sustain time could be made. Such a test with a physical tuned bar would be of great importance to musicians and researchers alike.

*Eric Laukkanen and Randy Worland (2011)*

Acoustical effect of progressive undercutting of percussive aluminum bars.

The research by Laukkanen represents the most recent work on this topic at the time of this writing. Similar to previous studies, finite element modeling and spectrum
analysis methods were employed. However the group also made notable use of electronic speckle pattern interferometry.

Where the two authors differ from previous research is in their search for a qualitative relationship regarding removal of mass from a given point about the length axis of the bar. Rather than studying continuous functions or piecewise functions across the span of the bar, Laukkanen and Worland produced FEA models of bars with relatively small cuts removed as shown in Figure 13. The relation $x/L$ was varied and the resulting frequencies of 1\textsuperscript{st}, 2\textsuperscript{nd}, and 3\textsuperscript{rd} Transverse modes are plotted in Figure 14. These plotted frequencies are normalized with the natural frequencies of an uncut uniform bar of the same width, length, and thickness. For reference, the dimensions for the bar studied by Laukkanen and Worland correspond to those typical of an F4 (~350 Hz) vibraphone bar.

![Figure 13 – Symmetric ½ thickness and 1/50 length cuts studied for effect on modal frequency.](image)
Similar to Orduna-Bustamante’s tuning curves, the above Figure 14 can be used as a visual guide and aid when hand tuning vibraphone bars. At the points of intersection, the frequency ratio of two modes can be held constant while the remaining mode can be tuned lower or higher depending whether the curve is above or below the point of intersection.

Experimental validation of the above methods included CAD and machining of beams with simple rectangular cuts as already described. The beams produced held harmonic ratios of 1:2:4 and 1:3:5. Upon spectral analysis, Laukkanen and Worland
found all three transverse mode frequencies agreed with the predicted values within 1% error. (Laukkanen & Worland, 2011)

The work by Laukkanen and Worland was a significant step forward in the field of visual aids for successful hand tuning of vibraphone bars. Furthermore, their methods along with Orduna-Bustamante’s methods can be employed to hand tune keys with non-traditional timbre. Although the accuracy of 1% is greatly beneficial for hand tuning aids further accuracy improvements would be necessary to eliminate hand tuning all together. Standard musical tolerances require 0.25% error. This is the motivation for the experimental work presented in the next section. If bars and keys could be machined with such accuracy, the cost of manufacturing and production would be drastically reduced.
COMPUTER ASSISTED DESIGN AND OPTIMIZATION OF BEAMS FOR USE IN MUSICAL VIBRAPHONE INSTRUMENTS

Abstract

Percussive idiophones such as the vibraphone have been produced by hand tuning since their invention. This entails an extensive manual labor process that is wasteful and expensive to the end user. The purpose of this research is to consider the application of finite element analysis and computer numerically controlled machining to the design and manufacture of bars with harmonic ratios (1:4:10) appropriate for vibraphones. Previous research has focused on using parabolic, Bezier, or piecewise functions to define an optimum profile, however most research neglects the impact of node holes. The finite element model was developed by optimizing an 11 knot cubic spline curve between the bar’s fundamental nodes. Other bar geometries such as length, width, and thickness were constrained to values appropriate for the instrument. Considering their end function, the modeling and machining of nodal instrument supports were considered in this research. All keys of the vibraphone instrument were successfully modeled and optimized to yield 1st, 2nd, and 3rd transverse frequencies of ±0.25% tolerance. Fundamental frequencies and overtones of two optimized keys were found to be within musical tolerances when tested via spectral analysis. Much of this work is now in patent application status (Kirkland, 2013).

PACS numbers: 43.40.CW, 43.10.KM
Research Design

Idiophones, percussive musical instruments such as marimbas, vibraphones, and xylophones, are among the world’s oldest instruments. Yet the keys of these instruments are still tuned as they were millennia ago, by arduously grinding their shape to achieve natural frequencies and harmonic overtones. The final product is a beautiful assembly of keys and resonators, increasing in popularity with orchestras. Despite this, strenuous production methods and resultantly high prices make the instruments unattainable for many schools and students.

Cost could be significantly reduced by defining optimum geometry before manufacturing rather than relying on trial and error. To this end, scientific inquiry has focused on mathematical descriptions of the keys, using parabolic, Bezier, or piecewise functions to define an optimum profile. To date, most research neglects the effect of node cuts (holes drilled to allow for strings on which the key will rest) which could have significant impact on the overtones.

This research models the keys of a vibraphone with Computer Assisted Design (CAD), and utilizes Finite Element Analysis (FEA) to refine and optimize their geometry with the inclusion of node cuts to achieve all the musically desired frequencies. These optimized structures can then be precisely machined, using sophisticated Computer Numerically Controlled (CNC) milling machines. If successful, the combined CAD and CNC manufacturing will result in an instrument tuned to within musical tolerances (± 5 cents) without the need for hand tuning. Additionally, a secondary goal of the design is to extend the range of a vibraphone by adding an extra octave, incorporating bass frequencies which have never existed on the instrument previously.
Before the methodologies are considered it is noted here that all modeling, FEA, and optimization described herein were performed using ProEngineer Creo and ProMechanica software.

**Topology Definition**

The most important geometric aspect considered in this paper is the undercut of each vibraphone key. This undercut will ultimately define the natural and harmonic frequencies of each key. Previous attempts at modeling this geometry have considered parabolic (Bustamante, 1991) and piecewise functions. Parabolas were modeled and optimized according to procedures outlined in the later sections of this paper. However they proved difficult to optimize for all target frequencies due to a limitation in variables to optimize (only depth and width of cut). Piecewise functions were proven by (Henrique, Inacio, Paulino, & Antunes, 2005) to result in accurately predicted frequencies, however there is an intrinsic difficulty in machining that would likely result in high cost of production.

In lieu of parabolic or piecewise functions, spline curves were chosen to model the undercut of the vibraphone keys. These curves ease machining by maintaining a gradually changing slope through the width of the cut. Further, any spline length can be defined by an infinite number of points resulting in many optimizable variables to achieve the target frequencies.

For this research presented here, the spline curves were broken into nine knots plus two knots at each end of the function. The two end knots were constrained to the lower surface at a distance of 0.244 times the length of the key, measured from the end of
the key. The nine remaining knots were equally spaced in the x dimension over the entire width of spline function. The y coordinate of each knot defined the cut depth at that particular point of the spline and were constrained to be symmetric about the center of the key. All these constraints together left five variables to be optimized, each defining the cut depth at their respective points. For clarification, a mathematical representation of the vibraphone key profile can be written as follows.

\[
y = \begin{cases} 
(\text{Thickness}) & \text{for } x < X_1 \text{ and } x \geq X_n \\
Y_i & \text{for } X_i \leq x < X_{i+1}
\end{cases}
\]

Where \(X_i\) is the x component of the \(i^{th}\) knot. \(n\) represents the total number of knots including the beginning and end points of the spline. \(Y_i\) is defined as follows and \(i\) represents the \(i^{th}\) piece of the cubic spline.

\[
Y_i = a_i + b_i t + c_i t^2 + d_i t^3
\]

\(0 \leq t \leq 1\)

By optimizing the y location of each knot of the spline curve, the following variables within the above equation are being constrained:

\[
Y_i(0) = a_i = y_i
\]

\[
Y_i(1) = a_i + b_i + c_i + d_i = y_{i+1}
\]

The variables \(b_i, c_i,\) and \(d_i,\) can be found by developing a system of equations based on the first and second derivative of the above two equations as detailed by (Bartels, Beatty, & Barsky, 1998).

The widths of the keys were constrained to typical values for a commercial vibraphone with graduating keys. The node holes were defined by the following process. First, the key was optimized without the presence of node holes and with a 1% tolerance on the target frequencies. A simple modal analysis was then done to find the locations of
the 1st transverse nodes. This was done for all keys of the instrument. With the locations of each node known for a given key a best fit line was generated. This line mathematically defined the locations of node holes for each key. Finally, with the node holes included in the CAD model, they key was reoptimized for 0.25% tolerance on the target frequencies.

Finally, the length of each key was investigated by two methods. First, length was considered as a sixth variable for optimization. As will be shown the results section, the resulting optimized keys would cause some significant design challenges. In response, the length was later specified as appropriate for a typical instrument and only the 5 spline variables were optimized.

To summarize and clarify, Table I lists the variables set/constrained and open for optimization.

<table>
<thead>
<tr>
<th>Constant</th>
<th>Variable Open to Optimization</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length*</td>
<td>$y_i$ for $1 &lt; i &lt; n$</td>
</tr>
<tr>
<td>Thickness</td>
<td></td>
</tr>
<tr>
<td>Width</td>
<td></td>
</tr>
<tr>
<td>$X_i$</td>
<td></td>
</tr>
<tr>
<td>$y_1, y_n$</td>
<td></td>
</tr>
</tbody>
</table>

Table I - *Note, Length was initially considered a variable however this caused some significant design problems as discussed in Results and Conclusions.
Meshing

Each beam of the vibraphone instrument was modeled with tetrahedral elements. Tetrahedrons were chosen for their improved accuracy in dynamic systems. Further, the tetrahedrons were constrained with a maximum edge length (node to node distance) of 5 mm. This resulted in models with as many as 10,000 elements depending on the length of key to be optimized. The keys were modeled as free, not fixed or pinned. This can be done with accuracy because the keys are supported through their node holes. Further, these holes typically have a large tolerance over the diameter of string to run through them. Because the node holes have differing angles and are not symmetric about the center, symmetric modeling and meshing cannot be employed. The effect of node holes on the frequencies of the keys is not clearly known by literature thus their inclusion is important.

Frequency Definition and Optimization

The variables defined above in the modeling section were identified to be optimized within the following lower and upper bounds:

\[ 0 < \text{cut depth} < \text{key thickness} \]

The target frequencies of each key were identified with a maximum tolerance of ±1.0% at first to identify node locations and then with ±0.25% tolerance after the node holes were incorporated. These target frequencies include the 1\(^{st}\), 2\(^{nd}\), and 3\(^{rd}\) transverse modes of vibration, all in ratios of 1:4:10. These ratios correspond to the typical timbre of a vibraphone. The fundamental frequencies were defined by the musical scale of equal
temperament with A4 equal to 442 Hz. Thus, these target frequencies in the case of the A4 bar may be written as nonlinear constraints defined as a general function as follows.

\[
\text{ceq} = f \begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 \\
y_5 \\
\end{pmatrix} - \begin{pmatrix}
442 \\
884 \\
4420 \\
\end{pmatrix}
\]

Equation 24

The Equation 24 shown above would be the format of a nonlinear constraint function written for optimization in MATLAB. Although the optimization algorithm was performed within ProEngineer, the function syntax is useful for explaining the constraint process. In this example, MATLAB would optimize the variables \(y_i\) to maintain \(\text{ceq} = 0\) within the 0.25\% tolerance. The function \(f(y_i)\) would perform a modal analysis of a beam with a spline curve defined by \(y_i\) and return resulting matrix of eigenvalues.

Sequential Quadratic Programming (SQP) was selected as the optimization algorithm for all keys. While maintaining the tolerances indicated above, the optimization goal was to minimize the total mass of the beam. Each key was re-meshed between each iteration of the algorithm while maintaining the 5 mm edge length defined previously.

**Verification of Computer Optimized Geometry**

The validity of the FEA model was experimentally validated by machining two copies of an A4 (442 Hz) key. These two keys were precision CNC machined according to the optimized CAD file geometry. The keys were then supported individually on strings as they would be if mounted on a commercial vibraphone. The keys were then struck using a standard commercial vibraphone mallet. Acoustic emission was recorded with a microphone and the data was imported into Audacity, an open source software
program capable of performing frequency spectral analysis under a Hamming window.
For this work, Hamming windowing functions were chosen because the exact frequencies within the spectrum are of much higher importance than the amplitude.

Results and Analysis

*Computational Results*

All 49 keys were successfully optimized to within the desired frequency tolerances. This includes the 37 traditional keys of a 3 octave instrument plus the extra 12 keys of the goal identified in the introduction.

Upon optimizing the length as well as the spline of each of the 49 keys, it becomes apparent the node holes will not fall close to a line but rather a curve. This is indicated by Figure 15. Note, in this figure, only the natural keys (those closest to the musician) are graphed.
Figure 15 – The resulting length of two methods graphed. Note, only natural keys are pictured.

As shown, the bass keys were optimized to be significantly longer than midrange and higher keys. In addition to the inherent difficulties of placing nodal supports on a curve, it was assumed such a design could encourage awkward playing habits in the musicians.

In response to the results shown in Figure 15, the length was constrained according to a logarithmic relationship. This relation was defined to prevent the keys on either end of the instrument from being too long or too short. As described in the research design section, the keys of constrained length were optimized to within 1% tolerance on the frequency constraints and the resulting nodes were identified. These nodes for the F Natural keys are shown in Figure 16. Once again, the typical range of a vibraphone covers F3 to F6. The optimized instrument was extended to F2 to F6, producing keys which are exceptionally uncommon to the instrument.
Figure 16 - Node 1 represents those closest to the musician. Node 2 represents those closer to the audience.

To further expose the difficulties of producing the optimized length design versus a constrained length approach, instrument drawings for both 4 octave vibraphones are shown below in Figure 17. The spacing between the keys shown in Figure 16 and Figure 17 assumes 10 mm.
Figure 17 – Drawing profiles for each instrument optimization. (Top) Length variable and optimized to reduce mass. (Bottom) Length constrained and only the spline optimized.
Experimental Measurement

To verify the computational models and optimizations, the optimized key A4 was selected to be machined. Two copies of the key were produced. The two machined keys are shown in Figure 18.

![Machined keys](image)

Figure 18 – Final machined key geometries. Both keys pictured were machined according to optimized parameters for an A4 vibraphone key.

Spectral analysis of the two machined samples revealed the following data presented in Table II. Recording of the keys was performed with a calibrated Quest Model 2700 Impulse Sound Level Meter. Bin size for the spectral analysis was set to 16384. Such a large bin size is necessary to yield as much measurement accuracy as possible. Because the keys have significantly little internal damping a large sample time
was possible, allowing for a large number of bins. Therefore, all measured frequencies are accurate within ±1.35 Hz.

<table>
<thead>
<tr>
<th>Sample Key</th>
<th>Mode Shape</th>
<th>Designed Frequency Hz</th>
<th>Measured Frequency ± 1.35 Hz</th>
<th>% Deviation from Design</th>
<th>Cents Error ε</th>
</tr>
</thead>
<tbody>
<tr>
<td>Key 1</td>
<td>1\textsuperscript{st} Transverse</td>
<td>442 Hz</td>
<td>441.43</td>
<td>0.13 %</td>
<td>-2</td>
</tr>
<tr>
<td></td>
<td>2\textsuperscript{nd} Transverse</td>
<td>1768 Hz</td>
<td>1773.79</td>
<td>0.28 %</td>
<td>+6</td>
</tr>
<tr>
<td></td>
<td>3\textsuperscript{rd} Transverse</td>
<td>4420 Hz</td>
<td>4422.38</td>
<td>0.05 %</td>
<td>+1</td>
</tr>
<tr>
<td>Key 2</td>
<td>1\textsuperscript{st} Transverse</td>
<td>442 Hz</td>
<td>441.43</td>
<td>0.13 %</td>
<td>-2</td>
</tr>
<tr>
<td></td>
<td>2\textsuperscript{nd} Transverse</td>
<td>1768 Hz</td>
<td>1773.79</td>
<td>0.28 %</td>
<td>+6</td>
</tr>
<tr>
<td></td>
<td>3\textsuperscript{rd} Transverse</td>
<td>4420 Hz</td>
<td>4422.38</td>
<td>0.05 %</td>
<td>+1</td>
</tr>
</tbody>
</table>

Table II
Figure 19 – Spectral frequency plot of key 1.

Figure 20 – Spectral frequency plot of bar 2.
Discussion

Because the harmonic ratio (1:4:10) was selected and achieved with little effort, this implies alternate ratios could be optimized just as easily. This statement is in agreement with the conclusions and findings of other literature in this area. Harmonic ratios are the most significant contributing factor of an instrument’s timbre, so optimizing custom harmonic ratios for unique instrument timbres is possible. Such instrument designs would likely be exceptionally difficult with current hand tuning methods due to unfamiliarity between traditional tuning professionals and the entirely new geometry which would be required.

The frequency spectrums of both manufactured and tested keys are remarkably similar with identical frequencies for the 1st, 2nd, and 3rd transverse modes. Importantly, this implies a significant degree of repeatability although the limited number of two keys is not a statistically significant sample size. Further research would benefit greatly from having a few machined keys from the low register of the instrument, but even more so from the high register. As shown by the previous literature, current methods in mathematical approximation of beam frequencies require use of either Bernoulli-Euler or Timoshenko Beam theories and assume all displacements are in a single plane. For short beams, shear and torsional moments become significant and the accuracy resulting from the single plane assumption suffers. Therefore, the ultimate test of tonal accuracy for musical applications requires testing of high frequency optimized beams. For example, the key F6 (1403 Hz) is typically the highest frequency on a standard vibraphone and the shortest physical length. It would make a great candidate for such testing.
Spectral graphs of both keys indicate a resonant frequency at approximately 885 Hz. This resonant frequency could correspond to a torsional mode of vibration. Some instrument manufacturers tune torsional modes as well as transverse. However most research papers in the field of mallet instrument optimization choose to ignore tuning and optimization of torsional beam modes. In general, this is a reasonable choice for the following reason. Proper technique when performing on any mallet instrument requires the musician to strike at the center of each bar or key. The first torsional mode of vibration has a large node located directly at that point of impact. This leads to minimal excitation of the first torsional mode. Therefore it acoustically contributes very little to the instrument’s timbre as perceived by the musician and audience.

However, neglecting the torsional mode and assuming the musician will always strike the center ignores the variability between techniques and individual musicians. In practical applications, striking a corner of the beam, and thus exciting a torsional mode is common.

One goal of the work presented here was to extend the traditional three octave (F3-F6) and additional octave (F2-F6). This was accomplished for all 49 keys of the larger instrument bandwidth. All keys of the instrument were successfully optimized using 1% frequency constraint tolerance. As shown by Figure 16, the resulting nodes were highly linear in nature. It is important to note, the linearity of the nodes are highly dependent on the given length of the keys. To a lesser extent, graduation on the width of the keys plays a part in any resulting linearity.

Continuing from the methods discussed in the previous chapter of this thesis, the goal of future research will be to assign a harmonic relationship to the first transverse
mode of vibration. The work by Rossing et al. should be used as a guide to determine the proper harmonic ratios for the torsional modes.

In summary, this original work has presented a basis for optimizing beam topology using target frequencies as constraints rather than arguments of an error function and instead optimizing another variable such as mass. Secondly, nodes were added to the models after optimizing with 1% tolerance on the constraints. These nodes were shown to be highly linear. However, this linearity is greatly dependent on predefined note length and key graduation. Finally, the key A4 with node holes was selected for physical machining and verification after optimizing with constraint tolerances of 0.25%. The final keys were found to be accurate to within standard musical tolerances.
FINAL CONCLUDING REMARKS

In this thesis work, a literature review of prior research into modeling of non-uniform beams for musical instrument applications has been presented. This prior modeling of beams has been shown to rely on traditional numeric methods based on Bernoulli-Euler or Timoshenko theories as well as FEA and FEM methods. In general, Timoshenko Beam theory has been shown by previous authors to be more accurate when modeling beams. This is because Timoshenko formulations include shear and rotary inertia which contribute greatly in shorter beams with higher fundamental frequencies.

Nearing the end of the 1990s decade, researchers began to turn to FEA techniques for accurate estimation of natural frequencies. This certainly owes in large part to the increase in computing power. For example, the work by Bork in 1999 utilized 2880 cubic elements and required several hours of CPU run time on a workstation. By contrast, today such calculations are routinely performed on laptop computers and the current work utilized as many as 10000 tetrahedral elements on a typical desktop computer for estimation of modal frequencies.

With FEA techniques quickly established as an accurate tool, many researchers such as Henrique et al. began parameterizing the geometric shape of the beams using piecewise, step, and continuous functions. The work which was presented here continues in that fashion, using cubic spline curves similar to the Bezier curves used by Henrique et al.
Again, similar to the work by Henrique et al. this thesis seeks to apply optimization techniques to the design of beams for musical applications. However a few significant differences are presented. Nodal support holes for final assembly on the instrument are rarely considered in literature. Here, those supports were implemented by performing rough optimization with 1% constraint tolerance, adding the support holes according to the found nodes, and optimizing again with a 0.25% constraint tolerance. Also, Henrique et al. performed their optimizations by minimizing the relative error of the frequencies. In this work the target frequencies were held as nonlinear constraints and the total mass of the beam was minimized using an SQP optimization algorithm.

Two beams optimized with the methods discussed were machined, both tuned to a fundamental frequency of 442 Hz. Spectral analysis of the keys showed strong resonance at the estimated modal frequencies. The measured frequencies of the machined keys were accurate within the very tight musical tolerances of 0.25%. This finding is significant for its applications to instrument manufacture as well as general design of structures with respect to eigenmodes and eigenfrequencies.

Further work in the design of beams for mallet percussion instruments should consider torsional and lateral modes of vibration. As shown by many authors, torsional modes in particular can color the perceived timbre and quality of the instrument. Tuning of these modes will require topology optimization of not just the undercut over the length of a beam but over the width as well.

Another aspect which needs to be investigated with further study is possible refinements to the FEA model. Free-free boundary conditions were applied to the beams
in this study. In reality, on the instrument the beams are supported by strings or wires at the node holes; this implies a boundary condition other than free is appropriate at the node locations. Generalization of the model to include this effect may be of little importance, since the node holes were placed at a location very close to the lowest fundamental mode of vibration, but due to the tight tolerances required for the beams, even a small computational error due to slightly inaccurate boundary conditions may be significant.

A future Ph.D. dissertation will apply the lessons learned in topology optimization for harmonic modes to the optimization of cavity resonance. The work by Bork and Henrique et al. already establishes the ground work for resonator optimization. However, it will be the goal of future research to expand on this by applying such optimization to the design of loudspeaker cabinets. It is hoped that the application of topology optimization may be able to disperse the resonance of loudspeaker enclosures to result in more linear signal generation.
SYSTEMS AND METHODS FOR DESIGNING AND FABRICATING MUSICAL INSTRUMENTS

WO2013172937A1

by

BRANDON KIRKLAND

APPLICANT

THE UAB RESEARCH FOUNDATION

Submitted to World Intellectual Property Organization
In preparation for Non-Provisional International Patent

Format adapted for thesis
Abstract

In one embodiment, a musical instrument component is designed by defining a
curvature of a surface of the component as a spline curve, setting target frequencies for
the component, and optimizing the spline curve so that the component produces the target
frequencies.

Description

SYSTEMS AND METHODS FOR
DESIGNING AND FABRICATING MUSICAL INSTRUMENTS

Cross-Reference to Related Application(s)

This application claims priority to co-pending U.S. Provisional Application serial number
61/647,783, filed May 16, 2012, which is hereby incorporated by reference herein in its
entirety.

Background

There are various instruments in the percussion family. The vibraphone is a
member of the mallet percussion family and uses rectangular aluminum keys having
grooves formed along their undersides that dictate the sound the keys make when they are
struck with a mallet. Specifically, the shape and depth of the grooves control the notes
that the keys make when struck.
Since their invention almost 200 years ago, vibraphone keys have been tuned by hand by grinding the groove into the underside of a key blank. The tuning process is laborious and typically takes many hours to complete for a single key. Moreover, if a mistake is made during the grinding process, such as grinding off too much of the key material, the key is ruined and the process must be started over with a new key blank.

In view of this, it can be appreciated that it would be desirable to have a more automated way of designing and fabricating instrument components, such as vibraphone keys.

_Brief Description of the Drawings_

The present disclosure may be better understood with reference to the following figures. Matching reference numerals designate corresponding parts throughout the figures, which are not necessarily drawn to scale.

Fig. 1 is a flow diagram of an embodiment of a method for designing an instrument component.

Fig. 2 is a perspective view of an example instrument key blank before a groove is formed in the blank to create a key.

Fig. 3 is a side view of a computer model of an instrument key illustrating a spline curve that defines the curvature of a groove that is to be formed in a key blank.

Fig. 4 is a perspective view of the instrument key model illustrating a mesh used to define the curve identified in Fig. 3.
Figs. 5A-5C are perspective views of the instrument key model of Fig. 4 that illustrate first, second, and third transverse modes of the key.

Fig. 6 is a perspective view of a fabricated instrument key.

Fig. 7 is a block diagram of an embodiment of a computing device that can be used to design an instrument component. Fig. 8 is a perspective view of a guitar sounding board that incorporates a bracing member having a surface whose shape has been designed using computer modeling.

Figs. 9A-9C are perspective views of a computer model of the sounding board of Fig. 8 that illustrate first, second, and third transverse modes of the sounding board.

*Detailed Description*

As described above, it would be desirable to have a more automated way of designing and fabricating instrument components, such as vibraphone keys. Described herein are systems and methods for designing and fabricating such components that are at least partially automated so that the component can be generated more quickly and easily, and with greater accuracy. In some embodiments, the systems and methods use computer-assisted design and finite element modeling to optimize the geometry of a surface of the component so that the components can be produced with a computer numerically controlled (CNC) process. The optimized geometry can, in some embodiments, achieve frequencies with overtones that are appropriate for an instrument, such as a vibraphone.
In the following disclosure, various specific embodiments are described. It is to be understood that those embodiments are example implementations of the disclosed inventions and that alternative embodiments are possible. All such embodiments are intended to fall within the scope of this disclosure.

As described above, hand tuning of keys for instruments such as vibraphones is laborious and time consuming. Part of the reason for the difficulty is that the instrument maker must not only achieve the correct fundamental frequency for the key, but must also achieve the correct overtones (i.e., higher modes of vibration), which are present in particular ratios to the fundamental frequency. As an example, vibraphone keys should resonate with the fundamental frequency and two associated overtones. If the correct overtones are not produced by the key when struck, the correct note may be sounded but the instrument will not have the proper timbre. Accordingly, the keys must be simultaneously tuned to produce the fundamental frequencies and their overtones.

It has been determined that the geometry needed to achieve the desired fundamental frequencies and overtones can be achieved with much less effort by optimizing the geometry of the keys using computer-aided design and finite element analysis and then shaping the key using a computer numerically controlled (CNC) process.

Fig. 1 describes an example process for the computer-aided design of an instrument component, such as a vibraphone key. Although vibraphone keys are repeatedly cited as an example in the discussion of Fig. 1 and other figures of this disclosure, it is noted that the disclosed principles can be applied to design and fabricate components of other instruments and sound-producing devices. Examples of such other instruments and
devices are discussed later in this disclosure. In some embodiments, the process described in relation to Fig. 1 can be performed using a computing device running commercially available computer-aided design software, such as ProEngineer Wildfire 5.0.

Beginning with block 10 of Fig. 1, the basic component geometry is set. As used herein, the term "set" means either selected or established by a user or by the software. In the case of a vibraphone key, the geometry of the key includes the length, width, and height dimensions of the key, which may be standardized. In some embodiments, the key is generally rectangular in cross-section and has a length dimension that is much larger than the width dimension, which is in turn larger than the height dimension. Fig. 2 illustrates an example key blank 30 prior to the formation of a groove on its underside. In some embodiments, the key blank 30 is made of aluminum.

Next, a curve that is to be used to define the curvature of the surface of the component (e.g., the groove on the underside of the key) is set, as indicated in block 12. In some embodiments, the curve is set to be a spline curve having multiple equidistant knots along its length. Such a configuration is illustrated in Fig. 3, which shows a computer model of a vibraphone key 40 in side view. In this example, the key 40 has a groove 42 that is defined by a spline curve 44 having 11 equally spaced knots 46. In some embodiments, a third-order cubic spline is used.

Spline curves can be used in the design process because they are particularly effective in performing interpolation between a set of points, such as the knots 46. Consider for example three points:
The spline curve between these points will be a piecewise curve. To find this curve, one first needs to find the slope at each of these points. This is accomplished by solving the matrix equation shown below.

\[
\begin{bmatrix}
  x_1 & y_1 \\
  x_2 & y_2 \\
  x_3 & y_3
\end{bmatrix} =
\begin{bmatrix}
  0 & 1 \\
  1 & 3 \\
  4 & -2
\end{bmatrix}
\]

For the three points above, this matrix equation equates to the following:

\[
\begin{bmatrix}
  a_{11} & a_{12} & 0 \\
  a_{12} & a_{22} & a_{23} \\
  0 & a_{23} & a_{33}
\end{bmatrix}
\begin{bmatrix}
  m_1 \\
  m_2 \\
  m_3
\end{bmatrix} =
\begin{bmatrix}
  b_1 \\
  b_2 \\
  b_3
\end{bmatrix}
\]

where: \( u = 2a \)

\( a_n - 2a_{23} \)

\( a_{22} = 2(a_{12} + a_{23}) \)

\[
\frac{3(y_2 - y_1)}{(x_2 - x_1)^2}
\]

\( b = 3 \)

\( (x_3 - x_2)^{3/4} = 3(i_1 + i_2) \)

For the three points above, this matrix equation equates to the following:
Solving for the slope matrix yields:

\[
\begin{bmatrix}
2 & 1 & 0 \\
1 & \frac{2}{3} & \frac{1}{3} \\
0 & \frac{1}{3} & \frac{2}{3}
\end{bmatrix}
\begin{bmatrix}
m_1 \\
m_2 \\
m_3
\end{bmatrix}
= 
\begin{bmatrix}
6 \\
13 \\
-\frac{12}{3}
\end{bmatrix}
\]

Solving for the slope matrix yields:

\[\text{[Equation 3]}\]

\[
\begin{bmatrix}
m_1 \\
m_2 \\
m_3
\end{bmatrix}
= 
\begin{bmatrix}
0.29167 \\
5.41667 \\
-5.20833
\end{bmatrix}
\]

Two new variables can be defined: \( C = m_1 (x_2 - x_1) - (y_2 - y_1) \) [Equation 4]

\( D = -m_2 (x_2 - x_1) + (y_2 - y_1) \) [Equation 5]

These variables can be utilized in the following parametric equation: \( q(t) = y_1 (1-t) + y_2 t + t (t-t) [C (t-t) + D t \text{ for } t \leq x \leq x_2] \) [Equation 6]

\( t \) [Equation 7]

The previous equation generates the interpolated curve from a point one to a point two. To obtain the second half of the curve, Equations 4, 5, 6, and 7 can be reapplied using coordinates and slopes for points two and three, instead of points one and two. After this
is completed and the algebra is simplified, the final equation can be represented by the piecewise function below.

[Equation 8]

\[
f(t) = \begin{cases} 
1 + 2t + t(1-t) \left[ -1.708(1-t) - 3.417t \right], & 0 \leq t \leq 1 \\
3(1-t) - 2t + t(1-t) \left[ 21.25(1-t) + 10.625t \right], & 0 \leq t \leq 1 
\end{cases}
\]

With reference back to Fig. 1 and block 14, the positions of the end knots 48 can be set. In the case of a vibraphone key, each end knot 48 can be, for example, spaced from an end of the key 40 a distance of approximately 0.244 of the length of the key. The end knots 48 have fixed y positions (depths) of zero and therefore lie within the plane of the planar underside of the key 40. Next, the initial positions of the other knots 46 are set, as indicated in block 16. In some embodiments, the initial y positions for the other knots 46 are selected by the user based on best guesses. Regardless, the y positions of the knots 46 will be adjusted during the optimization process. Optionally, the knots 46 can be constrained such that the curve 44 is symmetric about its center for aesthetic purposes.

Once the geometry of the component and its surface have been set, finite element analysis can be performed to optimize the shape of the spline curve to create the desired sound. Before this is accomplished, however, various parameters can be established. First, the properties of the material that are to be used to fabricate the component can be set, as indicated in block 18. In the case of a vibraphone, the material can be set to be aluminum, as is the norm. By way of example, the material can be set to be 606 -T6
aluminum. Of course, other materials could be used, such as another metal, wood, or a polymeric material.

Next, the maximum edge length of the mesh that will be used to define the shape of the surface can be set, as indicated in block 20. Example meshing of the key 40 is illustrated in Fig. 4. As is apparent from that figure, the mesh 50 includes multiple points 52. The maximum edge length is the maximum allowable distance between neighboring points 52 on the groove 42 of the key 40. In some embodiments, the maximum edge length can be set to approximately 5 or 6 millimeters (mm).

In addition, target frequencies can be set, as indicated in block 22. In some embodiments, the target frequencies are the fundamental frequency and two overtones so that there are three transverse modes. This includes the first transverse mode, which is the fundamental frequency, as well as a second transverse mode and a third transverse mode that are each defined by a harmonic ratio to the fundamental frequency. By way of example, the second transverse mode can have a harmonic ratio of 1 :4 to the fundamental frequency and the third transverse mode can have a harmonic ratio of 1 :10 to the fundamental frequency. In such a case, an A4 key with a fundamental frequency of 442 Hz would have a second transverse mode at 1768 Hz and a third transverse mode at 4420 Hz. Figs. 5A-5C illustrate example first, second, and third transverse modes, respectively, for the key 40.

At this point, an initial optimization can be performed, as indicated in block 24 of Fig. 1. In this optimization, the spline curve is optimized to generate the desired frequencies within the various predefined constraints, such as the component dimensions, materials,
etc. By way of example, the optimization can be performed with a tolerance for frequency of convergence of 1%. In some embodiments, the spline curve can be optimized to achieve one or more other desired qualities. For example, the spline curve can be optimized to minimize the mass, and therefore the weight, of the component. In cases in which the component is a key, the locations of its node holes, which are used to support the key in the instrument, can be set.

Referring next to block 26, the geometry of the spline curve can be updated based upon the initial optimization. If the component is a key, node holes can be added to the key model at the selected node locations. In such a case, a second optimization can be performed (block 28) to again optimize the spline curve, this time taking into account the presence of the node holes. If desired, the mass of the component can be minimized during the optimization process. By way of example, the optimization can be performed with a tolerance for frequency of convergence of 0.25%

After the second optimization, a completed design results. The completed design can be used as the input for a suitable high-precision fabrication machine, such as a CNC machine. In some embodiments, such a machine can achieve tolerances of 0.0001 inches or smaller. The machine can form the precise optimized shape of the surface of the component, such as the groove of the key. Optionally, surface treatments can then be added to the component, such as sandblasting or anodizing. Fig. 6 illustrates an example fabricated key 60 that includes an optimized groove 62 formed in its underside and node holes 64.
Fig. 7 illustrates an example configuration of a computing device 70 that can be used in the design process described above in relation to Figs. 1-5. As shown in the figure, the device 70 includes a processing device 72, memory 74, a user interface 76, and at least one I/O device 78, each of which is connected to a local interface 80.

The processing device 72 can include a central processing unit (CPU) or a semiconductor based microprocessor (in the form of a microchip). The memory 74 includes any one of or a combination of volatile memory elements (e.g., RAM) and nonvolatile memory elements (e.g., hard disk, ROM, etc.). The user interface 76 comprises the components with which a user interacts with the computing device 70, such as a keyboard and a display screen, and the I/O devices 78 are adapted to facilitate communications with other devices.

The memory 74 (a non-transitory computer-readable medium) comprises programs (i.e., logic) including an operating system 82, as well as a musical instrument component design system that includes or uses a computer-aided design program 84 and a finite element analysis program 86 that can be used to practice a method such as that described in relation to Fig. 1.

It is noted that the great precision of the systems and methods described herein may be utilized to achieve results that have not been achieved through hand manufacturing. For example, in the case of vibraphones, it may be possible to add an additional octave to the bass register of a vibraphone (i.e., the F₂-F₃ octave). Furthermore, because the keys can be optimized for specified harmonics, the keys can also be designed to sound like another instrument, such as a guitar, trumpet, etc.
As mentioned above, the principles disclosed herein are not limited to the design and fabrication of vibraphone keys. Therefore, these principles extend equally to the keys of other mallet percussion instruments, including xylophones, marimbas, and glockenspiels. Beyond this, however, the disclosed systems and methods can be used to design and fabricate components of instruments outside of the mallet percussion family. Indeed, the systems and methods can be used to design and fabricate the shapes of surfaces of any instrument component that has an effect on the sound that the instrument produces. Such components include the bracing and structural members of the instrument.

Fig. 8 illustrates an example of a non-percussion instrument that includes a component that can be designed and fabricated in the manner described above. More particularly, Fig. 8 shows a sounding board (i.e., front board) 90 of an acoustic guitar. Attached to an inner surface 92 of the sounding board 90 is a bracing member 94 that provides structural support to the sounding board and affects the sound that the guitar makes when played. In particular, the bracing member 94 provides color and tone (timbre) to the notes produced by the guitar and changes its sound quality.

As is apparent from Fig. 8, the bracing member 94, like the vibraphone key blank 30 shown in Fig. 2, has a generally rectangular cross-section. Similar to the fabricated key 60 shown in Fig. 6, the bracing member 94 has a curved surface 96 that can be designed using computer-aided design and spline curves as described in relation to Fig. 1. In particular, the shape of the surface 96 can be optimized using computer-aided design and finite element analysis to enable the sounding board 90 to produce a desired sound quality. As part of this process, the surface 96 of the member 94 can be optimized as
applied to the sounding board 90. Figs. 9A-9C illustrate example first, second, and third transverse modes, respectively, for the sounding board 90 with the bracing member 94 present.

The bracing or structural members of other instruments can be designed in a similar manner. For example, the bracing or structural members can be designed for acoustic basses, lyres, lutes, zithers, ukuleles, mandolins, pianos, harpsichords, bells, bowed string instruments (including violin, viola, cello, double bass), harps, cymbals, drums, and so forth.

The principles of this disclosure extend even beyond musical instruments. For example, the cabinet or enclosure of a speaker can be designed using the disclosed systems and methods. Indeed, substantially any component of any structure that has an effect on sounds that are produced by the structure can be designed and optimized using the disclosed systems and methods.

Claims

CLAIMS Claimed are:

1. A method for designing a component of a musical instrument, the method comprising:
   
   defining a curvature of a surface of the component as a spline curve;
   
   setting target frequencies for the component; and
   
   optimizing the spline curve so that the component produces the target frequencies.
2. The method of claim 1, wherein defining a curvature comprises defining the curvature as a third-order cubic spline.

3. The method of claim 1, wherein defining a curvature comprises setting the number of knots that the spline curve will have and initial positions for the knots.

4. The method of claim 1, wherein setting target frequencies comprises setting a fundamental frequency and one or more overtone frequencies.

5. The method of claim 1, wherein optimizing the spline curve comprises optimizing the curve to minimize a mass of the component.

6. The method of claim 1, wherein the component is a key of a percussion instrument.

7. The method of claim 1, wherein the component is a bracing or structural member of the instrument.

8. The method of claim 1, further comprising fabricating the component using a high-precision fabrication machine.

9. A system for designing a component of a musical instrument, the system comprising:

   a processing device; and

   memory that stores a musical instrument component design system comprising logic configured to:

   define a curvature of a surface of the component as a spline curve; set target frequencies for the component; and
optimize the spline curve so that the component produces the target frequencies.

10. The system of claim 9, wherein the logic configured to define a curvature is configured to define the curvature as a third-order cubic spline.

11. The system of claim 9, wherein logic configured to define a curvature is configured to set the number of knots that the spline curve will have and initial positions for the knots.

12. The system of claim 9, wherein logic configured to set target frequencies is configured to set a fundamental frequency and one or more overtone frequencies.

13. The system of claim 9, wherein logic configured to optimize the spline curve is configured to optimize the curve to minimize a mass of the component.

14. The system of claim 9, further comprising a high-precision fabrication machine that forms the surface of the component based on the optimized spline curve.

15. The system of claim 14, wherein the machine is a computer numerically controlled (CNC) machine.

16. A musical instrument comprising:

a component having a curved surface that has been designed using computer modeling and formed using a high-precision fabrication machine.

17. The instrument of claim 16, wherein the instrument is a mallet percussion instrument.

18. The instrument of claim 17, wherein the component is a key and the curved surface is formed on the underside of the key.
19. The instrument of claim 16, wherein the component is a bracing or structural member of the instrument.

20. The instrument of claim 19, wherein the instrument is a guitar.

<table>
<thead>
<tr>
<th>Cited Patent</th>
<th>Filing date</th>
<th>Publication date</th>
<th>Applicant</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>JP2001067067A</td>
<td></td>
<td></td>
<td>Marco Antonio Ferreira Cortes</td>
<td>Fretless grooved fingerboard</td>
</tr>
<tr>
<td>US6512168 *</td>
<td>Apr 20, 2001</td>
<td>Jan 28, 2003</td>
<td>Marco Antonio Ferreira Cortes</td>
<td>Tone plate for keyboard-type tone plate percussion instrument, tone plate fabricating method, tone generator unit of tone plate percussion instrument, and keyboard-type percussion instrument</td>
</tr>
<tr>
<td>US7541530 *</td>
<td>Dec 13, 2006</td>
<td>Jun 2, 2009</td>
<td>Yamaha Corporation</td>
<td>Milky bars arranged for percussion instruments</td>
</tr>
<tr>
<td>US20030167899 *</td>
<td>Sep 6, 2002</td>
<td>Sep 11, 2003</td>
<td>Hiroyasu Abe</td>
<td>Method for digitally copying parts of existing stringed</td>
</tr>
<tr>
<td>US20110137442 *</td>
<td>Dec 3, 2009</td>
<td>Jun 9, 2011</td>
<td>Waddle John R</td>
<td>Milky bars arranged for percussion instruments</td>
</tr>
<tr>
<td>Cited Patent</td>
<td>Filing date</td>
<td>Publication date</td>
<td>Applicant</td>
<td>Title</td>
</tr>
<tr>
<td>--------------</td>
<td>-------------</td>
<td>------------------</td>
<td>-----------</td>
<td>-------</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>musical instruments such as violins, violas, or cellos</td>
</tr>
</tbody>
</table>

* Cited by examiner

Classifications

<table>
<thead>
<tr>
<th>International Classification</th>
<th>G05B19/00, G10D13/00, G05B17/00</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cooperative Classification</td>
<td>G10D13/00, G05B19/00, G05B17/00, G10D13/08, G06F17/50</td>
</tr>
</tbody>
</table>
REFERENCES


