AMERICAN OPTIONS AND SEMILINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS IN WEIGHTED SOBOLEV SPACES.

by

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ABSTRACT

To value an American option as a function of time $t$ and price of the underlying asset $S$ is currently a major research problem in both Financial Markets and for academic purposes. Options and more general financial derivatives (also known as contingent claims) are now an important tool in risk management. One of the earliest models used in pricing derivatives is the Black-Scholes model, for which the movement in the price of the underlying asset on which the claim is based is modeled by geometric Brownian motion. Other models used for theoretical and numerical analysis of American options include the free boundary problem method, linear complimentarity problem method, and variational inequality methods. And there are others based on the Cox, Ross, and Rubinstein binomial approach.

Despite the existence of these methods, there is a strong practical demand to create new methods which firstly are more computationally efficient and make explicit the mathematical framework involved. Partial differential equations (PDE) of parabolic type have fundamental applications to modelling processes with diffusion and uncertainty. The pricing of European options can be reduced to the calculation of certain solutions of parabolic equations, often called backward Kolmogorov’s equations and obtained through Ito’s lemma. In this work we are dealing with American options and we transform the Black-Scholes equation into a nonlinear parabolic equation in the entire space variable,

\[
\begin{align*}
\begin{cases}
  u_t - u_{xx} = F(x,t,u), & \text{in } \mathbb{R} \times (0, \frac{a^2T}{2}) \\
  u(x,0) = u_0(x) \\
  |u(x,t)| \leq Ke^{c|x|}, & c > 0.
\end{cases}
\end{align*}
\]
The initial condition might be unbounded and so we strive to show the existence of the solution in the weighted Sobolev space $W^{1,2}_\lambda(\mathbb{R} \times (0, \frac{\sigma^2 T}{2}))$ for some $\lambda < 0$. 
DEDICATION

Dedicated to my Mother Kaleo Muthoka.
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I want to immensely thank my PhD adviser, Professor Marius Nkashama for allowing me to learn from him. He gave me the opportunity to study differential equations under his guidance. I am deeply greatful to him for his insights and advice that made studying mathematics the most enjoyable thing in my life. I thank all my PhD committee members because I have learned alot from them during my study in the Department and during the composition of the results in this work. I appreciate the faculty and staff at the Department of mathematics who were very friendly and created a very conducive environment for such scholarly work like this. I must also thank my fellow students in the Graduate program who were so helpful in so many ways during my study in the Department. Lastly but by no means least, I want to appreciate my family for being there for me even when I was not available for them.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>ii</td>
</tr>
<tr>
<td>DEDICATION</td>
<td>iv</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>v</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>vi</td>
</tr>
<tr>
<td><strong>CHAPTER 1. INTRODUCTION</strong></td>
<td>1</td>
</tr>
<tr>
<td>1. General Introduction</td>
<td>1</td>
</tr>
<tr>
<td>2. Derivation of the Black-Scholes Equation</td>
<td>3</td>
</tr>
<tr>
<td>3. European Options</td>
<td>7</td>
</tr>
<tr>
<td>4. American Options</td>
<td>10</td>
</tr>
<tr>
<td><strong>CHAPTER 2. MAXIMUM PRINCIPLES</strong></td>
<td>18</td>
</tr>
<tr>
<td>1. Unbounded Domains - Entire space</td>
<td>18</td>
</tr>
<tr>
<td><strong>CHAPTER 3. WEIGHTED SPACES</strong></td>
<td>23</td>
</tr>
<tr>
<td>1. Weighted Lebesgue Spaces</td>
<td>23</td>
</tr>
<tr>
<td>2. Weighted Sobolev Spaces</td>
<td>31</td>
</tr>
<tr>
<td>3. Special Weights</td>
<td>49</td>
</tr>
<tr>
<td><strong>CHAPTER 4. THE APPROXIMATE QUADRATIC SOLUTION</strong></td>
<td>56</td>
</tr>
<tr>
<td>1. The Linear problem</td>
<td>56</td>
</tr>
<tr>
<td>2. The Nonlinear problem</td>
<td>63</td>
</tr>
<tr>
<td>3. Conclusion</td>
<td>80</td>
</tr>
<tr>
<td><strong>CHAPTER 5. FUTURE DEVELOPMENTS - THE APPROXIMATE EXPONENTIAL SOLUTION</strong></td>
<td>81</td>
</tr>
<tr>
<td>References</td>
<td>90</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

3.1 The weight $w(x)$ for different values of $p > 0$. .......................... 26

3.2 The weight $w(x)^{-\frac{1}{p-1}}$ .................................................. 26

3.3 $w(x)$ for different values of $\lambda \geq 1$ ........................................ 32

3.4 $w(x)$ for different values of $0 < \lambda \leq 1$ ............................... 33

3.5 $w(x)$ for different values of $\lambda < 0$ ....................................... 33

3.6 The function $u(x)$ for different values of $\lambda \geq 1$ ..................... 34

3.7 $u(x)$ for different values of $0 < \lambda \leq 1$ ............................... 34

3.8 $u(x)$ for different values of $\lambda < 0$ ....................................... 35

3.9 The function $g_\delta(x)$ ............................................................ 36

4.1 The strip $\mathbb{R} \times (0, T_1)$ ..................................................... 67

4.2 The strips $\mathbb{R} \times (0, T_2)$ and $\mathbb{R} \times \left[T_1, \frac{\sigma T_2}{2}\right]$ .......... 71

4.3 The strip $\mathbb{R} \times (T_1, T_2)$ ..................................................... 72

4.4 The solution in $\mathbb{R} \times (0, T_1)$ .............................................. 73

4.5 The solution in $\mathbb{R} \times \left[T_1, \frac{3T_1}{2}\right]$ ................................ 74

4.6 The solution in $\mathbb{R} \times \left(0, \frac{3T_1}{2}\right)$ ................................ 74

4.7 The solution in $\mathbb{R} \times (T_1, 2T_1)$ ........................................ 75

4.8 The solution in $\mathbb{R} \times (0, 2T_1)$ ........................................... 75

4.9 The solution in $\mathbb{R} \times \left[\frac{3T_1}{2}, \frac{5T_1}{2}\right]$ ....................... 76

4.10 The solution in $\mathbb{R} \times \left(0, \frac{5T_1}{2}\right)$ ............................... 76

4.11 The solution in $\mathbb{R} \times \left(0, \frac{mT_1}{2}\right)$ ............................... 77
<table>
<thead>
<tr>
<th>Section</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.12</td>
<td>The solution in $\mathbb{R} \times \left( \frac{(m-1)T_1}{2}, \frac{(m+1)T_1}{2} \right)$</td>
<td>77</td>
</tr>
<tr>
<td>4.13</td>
<td>The solution in $\mathbb{R} \times \left( 0, \frac{(m+1)T_1}{2} \right)$</td>
<td>78</td>
</tr>
<tr>
<td>4.14</td>
<td>The solution in $\mathbb{R} \times \left( \frac{mT_1}{2}, \frac{\sigma^2 T_1}{2} \right)$</td>
<td>79</td>
</tr>
<tr>
<td>4.15</td>
<td>The solution in $\mathbb{R} \times \left( 0, \frac{\sigma^2 T_1}{2} \right)$</td>
<td>79</td>
</tr>
</tbody>
</table>
CHAPTER 1

INTRODUCTION

1. General Introduction

In chapter 1, we introduce the fundamental arguments that come from financial markets and that, coupled with techniques from mathematical analysis and probability theory, lead to the robust models that aptly describe the mathematical framework underlying financial markets. Of great significance here is Ito’s lemma that is actually the main ingredient in the derivation of Black-Scholes partial differential equation. We show that there is a serious difficulty that presents itself when the exercise time is not at the expiry time anymore but anytime before the expiry time. European options are shown to be easy to value and explicit formulas for the value of the European options are given. We show that American options are more complicated to value due to the fact that they can be exercised anytime before expiry. However in this work we spend a great deal of energy on the theoretical underpinnings, It must be noted that the theoretical work that we do here is eventually directed for purposes of explaining financial markets and that it is not for mere purposes of satisfying our curiosity.

In chapter 2, we present the maximum principles that will be needed in the rest of the work. Much has been done when it comes to maximum principles in bounded domains. Since our problem is cast on the entire space, we have to explore maximum principles on the entire space and we come up with a variation of the famous Phragmén-Lindelöf principle by way of changing the assumptions required in the Phragmén-Lindelöf principle for parabolic equations and still coming up with the same results see [49]. We come up with an important comparison theorem that we need. The ultimate goal is to establish some form of comparison theorem that enables us to define upper and lower solutions for the problem. The special case of the
Phragmén-Lindelöf principle is a new result and turns out to be completely useful not just in this work but even in other works where maximum principles are needed in the entire space. This leads to a comparison theorem for functions which are in weighted Sobolev spaces and this means that we have a comparison theorem for functions that might grow at infinity.

In chapter 3, we tackle the important topic of weighted spaces. We first begin by defining the weighted Lebesgue spaces. We have certain properties on the weight that make sure that the weighted Lebesgue spaces have some of the important properties of the classical Lebesgue spaces. It turns out that the weighted Lebesgue space is a Banach space regardless of the weight used. We show the properties that are needed on the weight in order that the functions in the weighted Lebesgue space can be viewed as distributions, also under certain mild conditions on the weight, the weighted Lebesgue spaces contain the set of test functions and the set of Schwartz functions as dense subsets. We then define the weighted Sobolev space, this is the space where we will need our solution to be sitting in. We show the clear distinction between the weighted and the classical Sobolev spaces. We expose the properties that we need on the weights in order to get some inclusions and compact imbeddings. It turns out that not every weight we use will give us a Banach space.

In chapter 4, we show the existence and uniqueness of the approximate solution to our nonlinear problem by first considering the linear problem. We show that if we fixed the unknown function on the source, we get a linear problem which we know has a unique solution in the appropriate weighted space. We are going to assume that the initial condition has a growth that is at most quadratic in the space variable, note that the initial condition is actually exponential in the space variable, so the assumption of it being quadratic is a rather strict one, but it fits in proper with the fixed point argument that we use to show the existence and uniqueness of a solution to our problem. We then show that the solution map defined this way has a fixed point
which is the solution to our nonlinear problem. This is the solution that we present in
the conclusion in chapter 4.

In chapter 5, We present the improvement that we intend to do on the solution.
Instead of the quadratic approximation that we have shown, we prove that we can get
upper and lower solutions which grow exponentially in the space variable, We show
that we have two sequences one increasing and another decreasing and we argue that
they converge to the same function and this is the solution to our problem.

2. Derivation of the Black-Scholes Equation

The return on an investment is defined to be the change in the price of the asset
divided by the original value. It’s a relative measure of the change and it’s a better
indicator than the absolute measure of the change. So if \( S \) is the asset price then
the return is given by \( \frac{dS}{S} \). How to model this return is a wide subject. One of the
common models decomposes the return into two parts, a predictable deterministic
part and a random stochastic part. The deterministic part is similar to the return on
money invested in a risk free bank and it gives the contribution \( \mu dt \) to the return,
\( \mu \) is a measure of the average rate of growth of the asset price and it’s also called
the drift, \( \mu \) can be taken to be constant or can also be viewed as a function of both
\( S \) and \( t \). The random component of the return is represented by a random sample
drawn from a normal distribution with mean zero and is given by \( \sigma dX \). \( \sigma \) is a number
called volatility and it measures the standard deviation of the returns. \( dX \sim N(0,dt) \),
\( \text{var}(X) = dt \). So then putting the two components together we get the stochastic
differential equation :

\[
\frac{dS}{S} = \sigma dX + \mu dt. \tag{1.1}
\]

Or if we let \( \sigma \) and \( \mu \) be functions of \( S \) and \( t \) we get the more complicated and
realistic model

\[
\frac{dS}{S} = \sigma(S,t)dX + \mu(S,t)dt. \tag{1.2}
\]
The term that contains the randomness is called a **Wiener process**. Since \( dX \sim N(0, dt) \), then we can choose \( \phi \sim N(0, 1) \) and define \( dX = \phi \sqrt{dt} \) and the stochastic differential equation is called a lognormal random walk because the probability density function of \( \frac{S+\Delta S}{S} \) follows a lognormal distribution. We will need the fact that \( dX^2 \to dt \) as \( dt \to 0 \) with probability 1 (see [64]), suppose \( V(S) \) is a smooth function of \( S \) and assume that \( S \) is not stochastic. If we vary \( S \) by a small amount \( dS \) then clearly \( V \) also varies by a small amount. Then using Taylor series we can expand:

\[
dV = \frac{dV}{dS} dS + \frac{1}{2} \frac{d^2V}{dS^2} dS^2 + ... \tag{1.3}
\]

but

\[
dS = \sigma S dX + \mu S dt,
\]

\[
dS^2 = \sigma^2 S^2 dX^2 + 2\sigma \mu S^2 dX dt + \mu^2 S^2 dt^2 \tag{1.4}
\]

and then we look at each term as \( dt \to 0 \). Since \( dX = O(\sqrt{dt}) \) then we get

\[
dS^2 \to \sigma^2 S^2 dt.
\]

We substitute this into the expression for \( dV \) and retain terms of order \( O(dt) \) and looking at \( V \) as a function of both \( S \) and \( t \) we get:

\[
dV = \frac{\partial V}{\partial S} (\sigma S dX + \mu S dt) + \frac{1}{2} \frac{\partial^2V}{\partial S^2} dS^2 + \frac{\partial V}{\partial t} dt.
\]

\[
dV = \sigma S \frac{\partial V}{\partial S} dX + \left( \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt. \tag{1.5}
\]

This is Itô's lemma which helps to relate the small change in a function of a random variable to the small change in the random variable itself. We can exploit the fact that the two random walks in \( S \) and \( V \) are both driven by the same random variable \( dX \) to eliminate the randomness. Let \( \gamma \) be any number and let \( W = V - \gamma S \) where \( \gamma \) is constant during the time step \( dt \), we then get:
\[ dW = dV - \gamma dS = \sigma S \frac{\partial V}{\partial S} dX + \left( \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt - \gamma \sigma S dX - \gamma \mu S dt \]

\[ = \sigma S \left( \frac{\partial V}{\partial S} - \gamma \right) dX + \left[ \mu S \left( \frac{\partial V}{\partial S} - \gamma \right) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right] dt. \]

We then choose \( \gamma = \frac{\partial V}{\partial S} \) and we get;

\[ dW = \left[ \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right] dt, \]

\[ \frac{dW}{dt} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}. \quad (1.6) \]

If we let \( V \) represent the value of an option - it can either be a put or a call option or an entire portfolio of different options, then we have seen by Ito’s lemma that:

\[ dV = \sigma S \frac{\partial V}{\partial S} dX + \left( \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt, \quad (1.7) \]

is the random walk followed by \( V \). We now construct a portfolio consisting of one option and a number \(-\Delta\) of the underlying asset, The value of the portfolio is,

\[ \Pi = V - \Delta S. \]

and the jump in one time step is

\[ d\Pi = dV - \Delta dS, \]

\[ d\Pi = \sigma S \frac{\partial V}{\partial S} dX + \left( \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt - \Delta \sigma S dX - \Delta \mu S dt \]

\[ = \sigma S \left( \frac{\partial V}{\partial S} - \Delta \right) dX + \left[ \mu S \left( \frac{\partial V}{\partial S} - \Delta \right) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right] dt. \quad (1.8) \]

We eliminate the randomness by letting \( \Delta = \frac{\partial V}{\partial S} \), we get a portfolio whose increment is wholly deterministic,

\[ d\Pi = \left[ \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right] dt. \quad (1.9) \]
Now using arguments from arbitrage and supply and demand, the return on a amount \( \Pi \) invested in a riskless assests would see a growth of \( r\Pi dt \) in a time interval \( dt \). If the right hand side of the equation above were greater than \( r\Pi dt \), an arbitrager could borrow an amount \( \Pi \) and invest in the portfolio and therefore make a guaranteed riskless profit. Conversely if the right hand side was less than \( r\Pi dt \), then the arbitrager could short the portfolio and invest \( \Pi \) in the bank. Either way the arbitrager would make a riskless, no cost, instantaneuos profit. The existence of such arbitragers with the ability to trade at low cost ensures that the return on the portfolio and on the riskless account are more or less equal:

\[
 r\Pi dt = \left[ -\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right] dt. \quad (1.10)
\]

but

\[
 \Pi = V - \frac{\partial V}{\partial S} S
\]

\[
rV dt - rS \frac{\partial V}{\partial S} dt = \left[ -\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right] dt. \quad (1.11)
\]

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \quad (1.12)
\]

This is the Black-Scholes partial differentail equation, which can be looked at from a European options or an American options point of view. For more details on this derivation see [37, 38, 63, 14, 38, 40, 41, 39, 40, 41, 42, 43, 44].
3. European Options

Let \( C(S, t) \) represent the value of a European call such that:

\[
\begin{align*}
\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + r S \frac{\partial C}{\partial S} - r C &= 0, \quad (0, \infty) \times (0, T) \\
C(0, t) &= 0, \\
C(S, t) &= S, \quad \text{as } S \to \infty, \\
C(S, T) &= \max(S - E, 0)
\end{align*}
\]

(1.13)

Assuming \( \sigma \) is constant, we then let \( S = E e^x \), \( t = T - \frac{2}{\sigma^2} \), \( C = \nu(x, \tau) \), then the equation we end up with is:

\[
\begin{align*}
\frac{\partial \nu}{\partial \tau} - \frac{\partial^2 \nu}{\partial x^2} &= \left( \frac{2r}{\sigma^2} - 1 \right) \frac{\partial \nu}{\partial x} - \frac{2r}{\sigma^2} \nu, \quad \text{in } \mathbb{R} \times (0, \frac{\sigma^2 T}{2}), \\
v(x, 0) &= \max\{(e^x - 1), 0\}, \\
v &\to 0, \text{ as } x \to -\infty, \\
v &\to \infty, \text{ as } x \to +\infty
\end{align*}
\]

(1.14)

Let \( k = \frac{2r}{\sigma^2} \) and assume \( \nu(x, \tau) = e^{\alpha x + \beta \tau} u(x, \tau) \) for some constants \( \alpha \) and \( \beta \) to be found and label \( \Psi(x, \tau) = e^{\alpha x + \beta \tau} \), then this equation:

\[
\begin{align*}
\frac{\partial \nu}{\partial \tau} - \frac{\partial^2 \nu}{\partial x^2} &= (k - 1) \frac{\partial \nu}{\partial x} - kv, \quad \text{in } \mathbb{R} \times (0, \frac{\sigma^2 T}{2}), \\
v(x, 0) &= \max\{(e^x - 1), 0\}, \\
v &\to 0, \text{ as } x \to -\infty, \\
v &\to \infty, \text{ as } x \to +\infty
\end{align*}
\]

(1.15)

becomes this:

\[
\begin{align*}
\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} &= 0, \quad \text{in } \mathbb{R} \times \left(0, \frac{\sigma^2 T}{2}\right) \\
u(x, 0) &= u_0(x) = \max\{(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}), 0\},
\end{align*}
\]

(1.16)
with

\[ \alpha = -\frac{k - 1}{2} = -\frac{1}{2}(k - 1) \]

\[ \beta = -\frac{1}{4}[k^2 + 2k - 1] = -\frac{1}{4}(k + 1)^2 \]

whose solution is given by see [33]

\[ u(x, \tau) = \frac{1}{2\sqrt{\pi \tau}} \int_{\mathbb{R}} u_0(s)e^{-\frac{(x-s)^2}{4\tau}} ds \]

because we know the specific function \( u_0(s) \), we can evaluate this integral to get,

\[ u(x, \tau) = e^{\frac{1}{4}(k+1)^2\tau + \frac{1}{2}(k+1)x} \Phi \left( \frac{x}{\sqrt{2\tau}} + \frac{(k+1)\sqrt{2\tau}}{2} \right) \]

\[ -e^{\frac{1}{4}(k-1)^2\tau + \frac{1}{2}(k-1)x} \Phi \left( \frac{x}{\sqrt{2\tau}} + \frac{(k-1)\sqrt{2\tau}}{2} \right) \]

where

\[ \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{s^2}{2}} ds. \] (1.17)

Then we recall that

\[ v(x, \tau) = e^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau} u(x, \tau) \]

and \( x = \log(S_E) \), \( \tau = \frac{\sigma^2}{2}(T-t) \), \( \sqrt{2\tau} = \sigma \sqrt{T-t} \), \( C = Ev(x, \tau) \) and so we can recover \( C \) by :

\[ C(S, t) = S\Phi \left( \frac{\log(S_E)}{\sigma \sqrt{T-t}} + \frac{(r + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}} \right) - Ee^{-r(T-t)}\Phi \left( \frac{\log(S_E)}{\sigma \sqrt{T-t}} + \frac{(r - \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}} \right). \] (1.18)

A similar argument can be presented for the European put option. The PDE governing the European put is given by :

\[
\begin{align*}
\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP &= 0, \quad (0, \infty) \times (0, T) \\
P(0, t) &= Ee^{-r(T-t)}, \\
P(S, t) &\rightarrow 0 \quad \text{as} \quad S \rightarrow \infty, \\
P(S, T) &= \max(E - S, 0)
\end{align*}
\] (1.19)
Then we let $S = E e^x$, $t = T - \frac{2\pi}{\sigma^2}$, $P = Ev(x, \tau)$, and $k = \frac{2\pi}{\sigma^2}$ then this is the equation we end up with:

\[
\begin{align*}
\frac{\partial v}{\partial \tau} - \frac{\partial^2 v}{\partial x^2} &= (k - 1)\frac{\partial v}{\partial x} - kv, & \text{in } \mathbb{R} \times (0, \frac{\sigma^2 T}{2}), \\
v(x, 0) &= \max(1 - e^x, 0), \\
v(x, \tau) &\to 0 \text{ as } x \to +\infty, \\
v(x, \tau) &\to E e^{-\frac{2\pi \tau}{\sigma^2}} \text{ as } x \to -\infty
\end{align*}
\]

(1.20)

we let $v(x, \tau) = u(x, \tau) \Psi(x, \tau)$, where $\Psi(x, \tau) = e^{\alpha x + \beta \tau}$ and $\alpha$ and $\beta$ are constants to be determined we get this equation:

\[
\begin{align*}
\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} &= 0, & \text{in } \mathbb{R} \times (0, \frac{\sigma^2 T}{2}), \\
u(x, 0) &= \max(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0) = u_0(x)
\end{align*}
\]

(1.21)

with $\alpha = -\frac{1}{2}(k - 1)$, $\beta = -\frac{1}{4}(k + 1)^2$

whose solution is see [33]

\[
u(x, \tau) = \frac{1}{2\sqrt{\pi \tau}} \int_{-\infty}^{+\infty} u_0(z) e^{-\frac{(z-x)^2}{4\tau}} dz
\]

Since we know the function $u_0(z)$ we can compute this integral to get,

\[
u(x, \tau) = e^{\frac{1}{2}(k-1)^2 \tau + \frac{1}{2}(k-1)x} \Phi\left(\frac{-x}{\sqrt{2\tau}} - \frac{(k - 1)\sqrt{2\tau}}{2}\right) - e^{\frac{1}{2}(k+1)^2 \tau + \frac{1}{2}(k+1)x} \Phi\left(\frac{-x}{\sqrt{2\tau}} - \frac{(k + 1)\sqrt{2\tau}}{2}\right)
\]

and then we recall that;

\[
u(x, \tau) = e^{-\frac{1}{2}(k-1)x - \frac{1}{2}(k+1)^2 \tau} u(x, \tau)
\]
\[ k = \frac{2r}{\sigma^2}, \tau = \frac{\sigma^2}{2}(T-t), x = \log(S/E), \sqrt{2\tau} = \sigma\sqrt{T-t}, P(S,t) = Ev(x,\tau) \]
to get;

\[ P(S,t) = Ee^{-r(T-t)}\Phi(-\frac{\log(S/E) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}) - S\Phi(-\frac{\log(S/E) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}). \]

As we will see in the next subsection, explicit formulas like the ones we have derived are very difficult to get when dealing with American options. A more detailed exposition of the mathematics behind European options can be found here ([41], [14], [37], [38], [39], [43], [45], [63], [44], [42]).

4. American Options

Elegant formulas are difficult to get when it comes to American options because American options can be exercised any time before the expiry time. The valuation of American options is therefore more complicated, since at each time we have to determine not only the option value, but also, for each value \( S \), whether or not it should be exercised -this constitutes a free boundary problem. The fact that we are able to exercise anytime before expiry depends on the assumption that there is a well defined pay-off for early exercise. This ability to exercise anytime makes American options potentially more valuable. Let \( V(S,t) \) be the value of the option and let \( g(S,t) \) be the well-defined pay-off, then for American options \( V(S,t) \geq g(S,t) \) unlike in the European options where this is sometimes violated. Let’s see why this is the case. If for instance \( V(S,t) < g(S,t) \), then one can buy the option at \( V(S,t) \) and immediately exercise it at \( g(S,t) \) and make an instantaneous risk free profit of \( g(S,t) - V(S,t) > 0 \). But of course such an opportunity would not last long before the value of the option is pushed up by the increased demand from arbitragers taking advantage of the market inconsistency. This means that there must be some values of \( S \) for which it’s optimal from the holder’s point of view to exercise the option because if this were not the case
then the option would have the European value since then Black-Scholes equation would hold for all $S$.

So at each time $t$ there is a particular value of $S$ which marks the boundary between two regions: to one side one should exercise the option and to the other side one should hold it, we denote this free boundary by $S_0(t)$. The arbitrage argument that gave the Black-Scholes equation for European options yields an inequality for American options. Since in the American options it is not necessarily possible for the option to be held long and short because there are times when it is optimal to exercise the option and so the writer can be exercised against, we can only say that the return from the portfolio can not be greater than the return from a bank deposit:

$$\frac{\partial v}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 v}{\partial S^2} + (r - D_0)S \frac{\partial v}{\partial S} - rv \leq 0$$

So we have established three important concepts for the American option which are:

- $v(S, t) \geq g(S, t)$
- There exists optimal price $S_0(t)$
- The Black-Scholes Equation is now an inequality.

When the price is not within the optimal range then we get

$$v(S, t) > g(S, t)$$

$$\frac{\partial v}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 v}{\partial S^2} + (r - D_0)S \frac{\partial v}{\partial S} - rv = 0$$

on the other hand if the price is within the optimal range then

$$\frac{\partial v}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 v}{\partial S^2} + (r - D_0)S \frac{\partial v}{\partial S} - rv < 0$$

$$v(S, t) = g(S, t)$$

in other words when the price is not within the optimal range we have

$$g(S, t) - v(S, t) < 0$$
\[
\frac{\partial v}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 v}{\partial S^2} + (r - D_0)S \frac{\partial v}{\partial S} - rv = 0,
\]

If the price is within the optimal range then we have:
\[
\frac{\partial v}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 v}{\partial S^2} + (r - D_0)S \frac{\partial v}{\partial S} - rv = \frac{\partial g}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 g}{\partial S^2} + (r - D_0)S \frac{\partial g}{\partial S} - rg < 0
\]

with
\[
g(S, t) - v(S, t) = 0
\]

If we define the Heaviside function by
\[
H(x) = \begin{cases} 
1, & \text{if } x \geq 0 \\
0, & \text{if } x < 0
\end{cases}
\]

Then we can write our equation like this
\[
\begin{cases} 
\frac{\partial v}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 v}{\partial S^2} + (r - D_0)S \frac{\partial v}{\partial S} - rv = \left(\frac{\partial g}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 g}{\partial S^2} + (r - D_0)S \frac{\partial g}{\partial S} - rg\right)H(g - v), \\
v(S, T) = g(S, T) \\
S \geq 0, \text{ and } t \in [0, T].
\end{cases}
\]

Let
\[
-\rho(S, t) = \frac{\partial g}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 g}{\partial S^2} + (r - D_0)S \frac{\partial g}{\partial S} - rg < 0
\]

This is called the loss flow and
\[
\rho(S, t) = -\left(\frac{\partial g}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 g}{\partial S^2} + (r - D_0)S \frac{\partial g}{\partial S} - rg\right) > 0
\]

is called the cash flow. Then our equation looks like this
\[
\begin{cases} 
\frac{\partial v}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 v}{\partial S^2} + (r - D_0)S \frac{\partial v}{\partial S} - rv = -\rho(S, t)H(g - v), & \text{in } (0, \infty) \times (0, T) \\
v(S, T) = g(S, T)
\end{cases}
\]
We then transform this into a parabolic equation by letting \( x = \log S \), \( \tau = \frac{\sigma^2}{2} (T - t) \)

\[ v(S, t) = e^{ax - \beta \tau} u(x, \tau) \]

then \( \frac{\partial x}{\partial S} = \frac{1}{S}, \ \frac{\partial \tau}{\partial t} = -\frac{\sigma^2}{2} \). Let \( \Psi(x, \tau) = e^{ax - \beta \tau} \) then

\[ v = u \Psi \]  
(1.28)

\[ \frac{\partial v}{\partial t} = (-\beta u \Psi + \frac{\partial u}{\partial \tau} \Psi)(-\frac{\sigma^2}{2}) \]  
(1.29)

\[ \frac{\partial v}{\partial S} = (u \Psi_x + u_x \Psi) \frac{\partial x}{\partial S} = (au \Psi + u_x \Psi) \frac{1}{S} \]

\[ \frac{\partial^2 v}{\partial S^2} = \left[ a(u \Psi_x \frac{\partial x}{\partial S} + u_x \Psi \frac{\partial x}{\partial S}) + (u \Psi_x \frac{\partial x}{\partial S} + u_{xx} \Psi \frac{\partial x}{\partial S}) \right] \frac{1}{S} - \frac{1}{S^2} (au \Psi + u_x \Psi) \]

\[ \frac{\partial^2 v}{\partial S^2} = \left[ a^2 u \Psi + au_x \Psi + au_x \Psi + u_{xx} \Psi \right] \frac{1}{S} - \frac{1}{S^2} (au \Psi + u_x \Psi) \]

\[ \frac{\partial^2 v}{\partial S^2} = \left[ a^2 u + au_x + au_x + u_{xx} \right] \frac{\Psi}{S^2} - \frac{\Psi}{S^2} (au + u_x) \]

\[ S^2 \frac{\partial^2 v}{\partial S^2} = [a^2 u + 2au_x + u_{xx}] \Psi - \Psi (au + u_x) \]

\[ S^2 \frac{\partial^2 v}{\partial S^2} = [(a^2 - a)u + (2a - 1)u_x + u_{xx}] \Psi \]

\[ \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} = \frac{\sigma^2}{2} [(a^2 - a)u + (2a - 1)u_x + u_{xx}] \Psi \]  
(1.30)
\[ (r - D_0)S \frac{\partial v}{\partial S} = (r - D_0)(\alpha u + u_x)\Psi \]  

(1.31)

\[ rv = ru \Psi \]  

(1.32)

So then our transformed equation looks like this:

\[
\frac{\partial v}{\partial t} + \frac{\sigma^2 S^2 \partial^2 v}{2 \partial S^2} + (r - D_0)S \frac{\partial v}{\partial S} - rv = \left( -\beta u \Psi + \frac{\partial u}{\partial \tau} \Psi \right) \left( -\frac{\sigma^2}{2} \right) \\
+ \frac{\sigma^2}{2} \left[ (\alpha^2 - \alpha)u + (2\alpha - 1)u_x + u_{xx} \right] \Psi \\
+ (r - D_0)(\alpha u + u_x)\Psi - ru \Psi
\]

\[
= \left[ \frac{\sigma^2}{2} \beta u - \frac{\sigma^2}{2} u_x + \frac{\sigma^2}{2} (\alpha^2 - \alpha)u + \frac{\sigma^2}{2} (2\alpha - 1)u_x + \frac{\sigma^2}{2} u_{xx} + (r - D_0)\alpha u + (r - D_0)u_x - ru \right] \Psi
\]

\[
= -\frac{\sigma^2}{2} (u_\tau - u_{xx}) \Psi + \left[ \frac{\sigma^2}{2} \beta + \frac{\sigma^2}{2} (\alpha^2 - \alpha) + (r - D_0)\alpha - r \right] u \Psi + \left[ \frac{\sigma^2}{2} (2\alpha - 1) + (r - D_0) \right] u_x \Psi
\]

let

\[
\frac{\sigma^2}{2} \beta + \frac{\sigma^2}{2} (\alpha^2 - \alpha) + (r - D_0)\alpha - r = 0
\]

and

\[
\frac{\sigma^2}{2} (2\alpha - 1) + (r - D_0) = 0
\]

and solve for \( \alpha \) and \( \beta \) to get:

\[
\alpha = -\frac{1}{2} \left[ \frac{2}{\sigma^2(r-D_0) - 1} \right]
\]

and

\[
\beta = \frac{1}{4} \left[ 1 - \frac{2}{\sigma^2(r-D_0)} \right]^2 + \frac{2r}{\sigma^2} > 0
\]

Since we know that during exercise
\[ v(S,t) = g(S,t) \] but \[ v(x,\tau) = u(x,\tau)e^{ax-\beta\tau} = g(S,t) = g(e^x,T - \frac{2\tau}{\sigma^2}) \], so at exercise

\[ u(x,\tau) = g(e^x,T - \frac{2\tau}{\sigma^2})e^{-ax+\beta\tau}. \]

Then define

\[ g(x,\tau) = g(e^x,T - \frac{2\tau}{\sigma^2})e^{-ax+\beta\tau} \tag{1.33} \]

\[ g = \hat{g}\Psi. \]

This implies

\[ \frac{\partial g}{\partial t} = (-\beta \hat{g}\Psi + \frac{\partial \hat{g}}{\partial \tau}\Psi)(-\frac{\sigma^2}{2}) \tag{1.34} \]

\[ \frac{\partial g}{\partial S} = (\hat{g}\Psi_x + \hat{g}_x\Psi)\frac{\partial x}{\partial S} = (a \hat{g}\Psi + \hat{g}_x\Psi)\frac{1}{S} \]

\[ \frac{\partial^2 g}{\partial S^2} = [a(\hat{g}\Psi_x + \hat{g}_x\Psi)\frac{1}{S} + (a \hat{g}_x\Psi + \hat{g}_{xx}\Psi)\frac{1}{S}]\frac{1}{S} - \frac{1}{S^2}(a \hat{g}\Psi + \hat{g}_x\Psi) \]

\[ \frac{\partial^2 g}{\partial S^2} = [a^2 \hat{g}\Psi + a \hat{g}_x\Psi + a \hat{g}_x\Psi + \hat{g}_{xx}\Psi]\frac{1}{S^2} - \frac{1}{S^2}(a \hat{g}\Psi + \hat{g}_x\Psi) \]

\[ \frac{\partial^2 g}{\partial S^2} = [a^2 \hat{g}\Psi + a \hat{g}_x\Psi + a \hat{g}_x\Psi + \hat{g}_{xx}\Psi]\frac{1}{S^2} - \frac{1}{S^2}(a \hat{g}\Psi + \hat{g}_x\Psi) \]

\[ \frac{\partial^2 g}{\partial S^2} = [a^2 \hat{g} + a \hat{g}_x + a \hat{g}_x + \hat{g}_{xx}]\Psi - \frac{1}{S^2}(a \hat{g} + \hat{g}_x) \]

\[ S^2 \frac{\partial^2 g}{\partial S^2} = [a^2 \hat{g} + 2a \hat{g}_x + \hat{g}_{xx}]\Psi - \Psi(a \hat{g} + \hat{g}_x) \]

\[ S^2 \frac{\partial^2 g}{\partial S^2} = [(a^2 - a)\hat{g} + (2a - 1)\hat{g}_x + \hat{g}_{xx}]\Psi \]
\[
\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 g}{\partial S^2} = \frac{\sigma^2}{2}[(\alpha^2 - \alpha) \hat{g} + (2\alpha - 1) \hat{g}_x + \hat{g}_{xx}]\Psi
\]  

(1.35)

\[
(r - D_0)S \frac{\partial g}{\partial S} = (r - D_0)(a \hat{g} + \hat{g}_x)\Psi
\]  

(1.36)

\[
rg = r \hat{g} \Psi
\]  

(1.37)

So then our transformed equation looks like this:

\[
\frac{\partial g}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 g}{\partial S^2} + (r - D_0)S \frac{\partial g}{\partial S} - rg = (-\beta \hat{g} \Psi + \frac{\partial \hat{g}}{\partial \tau} \Psi)(-\frac{\sigma^2}{2}) + \frac{\sigma^2}{2}[(\alpha^2 - \alpha) \hat{g} + (2\alpha - 1) \hat{g}_x + \hat{g}_{xx}]\Psi
\]

\[
+ (r - D_0)(a \hat{g} + \hat{g}_x)\Psi - r \hat{g} \Psi
\]

\[
= \frac{\sigma^2}{2} \beta \hat{g} - \frac{\sigma^2}{2} \hat{g}_\tau + \frac{\sigma^2}{2}(\alpha^2 - \alpha) \hat{g} + \frac{\sigma^2}{2}(2\alpha - 1) \hat{g}_x + \frac{\sigma^2}{2} \hat{g}_{xx} + (r - D_0)a \hat{g} + (r - D_0)\hat{g}_x - r \hat{g}\Psi
\]

\[
= \frac{-\sigma^2}{2}(\hat{g}_\tau - \hat{g}_{xx})\Psi + \left[\frac{\sigma^2}{2}\beta + \frac{\sigma^2}{2}(\alpha^2 - \alpha) + (r - D_0)\alpha - r\right] \hat{g} \Psi + \left[\frac{\sigma^2}{2}(2\alpha - 1) + (r - D_0)\right] \hat{g}_x \Psi
\]

but we already know that

\[
\frac{\sigma^2}{2} \beta + \frac{\sigma^2}{2}(\alpha^2 - \alpha) + (r - D_0)\alpha - r = 0
\]

and

\[
\frac{\sigma^2}{2}(2\alpha - 1) + (r - D_0) = 0
\]

so we get:

\[-\rho(S, t) = \frac{-\sigma^2}{2}(\hat{g}_\tau - \hat{g}_{xx})\Psi\]

and our equation becomes
\[-\frac{\sigma^2}{2}(u_\tau - u_{xx})\Psi = \frac{\sigma^2}{2} (\hat{g}_\tau - \hat{g}_{xx})\Psi H(g(e^x, T - \frac{2\tau}{\sigma^2})e^{-ax + \beta \tau} - u(x, \tau))\]

\[u_{\tau} - u_{xx} = -(\hat{g}_\tau - \hat{g}_{xx})H(\hat{g}(x, \tau) - u(x, \tau)).\]

Define \(\hat{\rho}(x, \tau) = \max((\hat{g}_\tau - \hat{g}_{xx}), 0) \geq 0\).

So this is the final equation that we will study:

\[
\begin{cases}
  u_\tau - u_{xx} = -\hat{\rho}(x, \tau)H(\hat{g}(x, \tau) - u(x, \tau)), \quad \text{in } \mathbb{R} \times (0, \frac{\sigma^2 T}{2}) \\
  u(x, 0) = \hat{g}(x, 0) = g(e^x, T)e^{-ax}
\end{cases}
\]

(1.38)

We show the existence of a unique solution to an approximate problem in some appropriate weighted Sobolev space after assuming that \(\hat{\rho}\) is bounded. To do this we need some preliminary results and in the next two chapters we present the preliminary tools that we will need to show the existence and uniqueness of the solution. For more information on American options see (\([45, 39, 43, 14, 63, 38, 37]\).)
CHAPTER 2

MAXIMUM PRINCIPLES

Since we will be using the concept of upper and lower solutions, we will need some comparison theorem for functions defined on the entire space. Since maximum principles on the entire space are much more delicate than maximum principles on bounded domains with some described boundary conditions, we will spend a great deal of our time trying to clearly expose the maximum principle arguments needed in order to get the comparison theorem for functions in the entire space. We modify the famous Phragmén- Lindelöf principle for real variables - change its assumptions a little - and then we show how to come up with the same comparison results. In particular we assume that the symmetric matrix $a^{ij}$ has each of its elements positive, this is a new assumption as opposed to the original Phragmén-Lindelöf principle which assumes some kind of decay as you go further out. More on this argument see [49, 29, 35, 36, 32, 33].

1. Unbounded Domains - Entire space

THEOREM 2.1. (Phragmén-Lindelöf principle)

Given

$$
\begin{cases}
-u_t + \sum_{i,j} a^{ij} u_{x_i x_j} + hu \geq 0, & \text{in } \mathbb{R}^d \times (0, T) \\
 u(x, 0) \leq 0, \\
\text{for } c > 0, |u(x, t)| \leq Ke^{c|x|^2} \text{ uniformly in } t
\end{cases}
$$

(2.1)

$a^{ij} \geq 0, \ a^{ij}, h \in L^\infty(\mathbb{R}^d \times (0, T)), \ m|\xi|^2 \leq \sum_{i,j} a^{ij} \xi_i \xi_j \leq M|\xi|^2 \text{ for every } \xi \in \mathbb{R}^d, \text{ then}$

$$u(x, t) \leq 0, \text{ in } \mathbb{R}^d \times (0, T)$$
PROOF. Let \( \Psi(x,t) = e^{-\frac{c|x|^2}{\gamma - ct} - \beta t} \) for \( \gamma > 0 \) and \( \beta > 0 \) to be determined, and let \( v(x,t) = u(x,t)\Psi(x,t) \). Choose \( \gamma - ct > 0 \) so that \( 0 < t < \frac{\gamma}{c} \) then this choice ensures that \( v(x,t) \to 0 \) uniformly as \( |x| \to \infty \). \( \Psi_t = -(\frac{c^2\gamma|x|^2}{(\gamma - ct)^2} + \beta)\Psi \) and \( \Psi_{x_i} = -\frac{2c\gamma}{\gamma - ct}x_i u\Psi \),

\[ v_t = u_t \Psi - (\frac{c^2\gamma|x|^2}{(\gamma - ct)^2} + \beta)u\Psi, \]

\[ v_{x_i} = u_{x_i} \Psi - \frac{2c\gamma}{\gamma - ct}x_i u\Psi. \]

\[
v_{x_i,x_j} = u_{x_i,x_j} \Psi - \frac{4c\gamma}{\gamma - ct}x_j u_{x_i} \Psi - \frac{2c\gamma}{\gamma - ct}(x_i)_j u \Psi + \frac{4c^2\gamma^2}{(\gamma - ct)^2}x_ix_j u \Psi. \]

\[
-v_t + \sum_{i,j} a^{ij} v_{x_i,x_j} + hv = -u_t \Psi + (\frac{c^2\gamma|x|^2}{(\gamma - ct)^2} + \beta)u\Psi + \sum_{i,j} a^{ij} u_{x_i,x_j} \Psi - \frac{4c\gamma}{\gamma - ct} \sum_{i,j} a^{ij}x_j u_{x_i} \Psi - \frac{2c^2\gamma^2}{(\gamma - ct)^2} \sum_{i,j} a^{ij} x_i x_j u \Psi + hu \Psi. \]

\[
-v_t + \sum_{i,j} a^{ij} v_{x_i,x_j} + hv = -u_t \Psi + (\frac{c^2\gamma|x|^2}{(\gamma - ct)^2} + \beta)u\Psi + \sum_{i,j} a^{ij} u_{x_i,x_j} \Psi - \frac{4c\gamma}{\gamma - ct} \sum_{i,j} a^{ij}x_j u_{x_i} \Psi - \frac{2c^2\gamma^2}{(\gamma - ct)^2} \sum_{i,j} a^{ij} x_i x_j u \Psi + hu \Psi. \]

\[
-v_t + \sum_{i,j} a^{ij} v_{x_i,x_j} + hv = (u_t + \sum_{i,j} a^{ij} u_{x_i,x_j} + hu) - \frac{4c\gamma}{\gamma - ct} \sum_{i,j} a^{ij}x_j u_{x_i} \Psi - \frac{8c^2\gamma^2}{(\gamma - ct)^2} \sum_{i,j} a^{ij} x_i x_j u \Psi - \frac{4c^2\gamma^2}{(\gamma - ct)^2} \sum_{i,j} a^{ij} x_i x_j u \Psi - \frac{2c\gamma}{\gamma - ct} \sum_{i,j} a^{ij} u_{x_i,x_j} \Psi + (\frac{c^2\gamma|x|^2}{(\gamma - ct)^2} + \beta)u\Psi. \]

\[
-v_t + \sum_{i,j} a^{ij} v_{x_i,x_j} + hv = (u_t + \sum_{i,j} a^{ij} u_{x_i,x_j} + hu) - \frac{4c\gamma}{\gamma - ct} \sum_{i,j} a^{ij}x_j u_{x_i} \Psi - \frac{8c^2\gamma^2}{(\gamma - ct)^2} \sum_{i,j} a^{ij} x_i x_j u \Psi - \frac{4c^2\gamma^2}{(\gamma - ct)^2} \sum_{i,j} a^{ij} x_i x_j u \Psi - \frac{2c\gamma}{\gamma - ct} \sum_{i,j} a^{ij} u_{x_i,x_j} \Psi + (\frac{c^2\gamma|x|^2}{(\gamma - ct)^2} + \beta)u\Psi. \]

\[
[- \frac{4c^2\gamma^2}{(\gamma - ct)^2} \sum_{i,j} a^{ij} x_i x_j - \frac{2c\gamma}{\gamma - ct} \sum_{i,j} a^{ij} + \frac{c^2\gamma|x|^2}{(\gamma - ct)^2} + \beta]u\Psi. \]
\[-v_t + \sum_{i,j}^d a_{ij} v_{x_i x_j} + hv = \Psi(-u_t + \sum_{i,j}^d a_{ij} u_{x_i x_j} + hu) - \frac{4 \epsilon^c}{\gamma - ct} \sum_{i,j}^d a_{ij} x_i v_{x_j} + \frac{1}{(\gamma - ct)^2} \sum_{i,j}^d a_{ij} x_i x_j - \frac{2 \epsilon^c}{\gamma - ct} \sum_{i,j}^d a_{ii} - \frac{c^2 \gamma |x|^2}{(\gamma - ct)^2} - \frac{\beta}{u} \Psi. \Rightarrow\]

\[
\frac{2 \epsilon^c}{\gamma - ct} \sum_{i,j}^d a_{ii} + \frac{c^2 \gamma |x|^2}{(\gamma - ct)^2} + \beta |u| \Psi. \Rightarrow
\]

\[
\Psi(-u_t + \sum_{i,j}^d a_{ij} u_{x_i x_j} + hu) = -v_t + \sum_{i,j}^d a_{ij} v_{x_i x_j} + hv + \frac{4 \epsilon^c}{\gamma - ct} \sum_{i,j}^d a_{ij} x_i v_{x_j} + \frac{1}{(\gamma - ct)^2} \sum_{i,j}^d a_{ij} x_i x_j + \frac{2 \epsilon^c}{\gamma - ct} \sum_{i,j}^d a_{ii} - \frac{c^2 \gamma |x|^2}{(\gamma - ct)^2} - \beta + h |v. \Rightarrow
\]

let \( H(x, t) = \frac{4 \epsilon^c}{(\gamma - ct)^2} \sum_{i,j}^d a_{ij} x_i x_j + \frac{2 \epsilon^c}{\gamma - ct} \sum_{i,j}^d a_{ii} - \frac{c^2 \gamma |x|^2}{(\gamma - ct)^2} - \beta + h. \)

\[
H(x, t) \leq \frac{4 \epsilon^c}{(\gamma - ct)^2} M |x|^2 + \frac{2 \epsilon^c}{\gamma - ct} \sum_{i,j}^d a_{ii} - \frac{c^2 \gamma |x|^2}{(\gamma - ct)^2} - \beta + h.
\]

\[
H(x, t) \leq \frac{2 \epsilon^c}{\gamma - ct} \sum_{i,j}^d a_{ii} - \frac{c^2 \gamma |x|^2}{(\gamma - ct)^2} [1 - 4M \gamma] - \beta + h.
\]
\[ H(x,t) \leq -\frac{c\gamma |x|^2}{(\gamma - ct)^2} [1 - 4M\gamma] + \left( \frac{2c\gamma}{\gamma - ct} \sum_{i,j} a^{ij} \right) - \beta + h. \]

If we choose \( \gamma < \min \left\{ \frac{1}{4M}, cT \right\} \) and \( \beta > \frac{2c\gamma}{\gamma - ct} \sum_{i,j} a^{ij} + h \), we ensure that \( H(x,t) \leq 0 \) and so we have a new problem which looks like this

\[
\begin{cases}
-v_t + \sum_{i,j} a^{ij} v_{x_i x_j} + \frac{4c\gamma}{\gamma - ct} \sum_{i,j} a^{ij} x_i v_{x_j} + H(x,t)v \geq 0, & \text{in } \mathbb{R}^d \times (0, \gamma) \\
v(x,0) \leq 0 & \\
v(x,t) \to 0, & \text{as } |x| \to \infty
\end{cases}
\]  

(2.2)

then for any \( R > 0 \), \( v \) can not attain a positive maximum at an interior point in \( \{|x| < R\} \times (0, \gamma) \) because this would contradict the inequality. So for any \( \epsilon > 0 \), \( v < \epsilon \) on \( \{|x| = R\} \times (0, \gamma) \) for some arbitrarily large \( R \), and since \( v(x,0) \leq 0 \), it follows that \( v \leq \epsilon \) in \( \{|x| < R\} \times (0, \gamma) \). Letting \( \epsilon \to 0 \) and \( R \to \infty \) we get that \( v \leq 0 \) in \( \mathbb{R}^d \times (0, \gamma) \). The entire argument can be repeated with \( t = \gamma/c \) as the initial surface instead of \( t = 0 \), in this way we obtain \( v \leq 0 \) in \( \mathbb{R}^d \times (\gamma/c, 2\gamma/c) \). In a finite number of steps we obtain \( v \leq 0 \) in \( \mathbb{R}^d \times (0, T) \) and hence \( u \leq 0 \) in \( \mathbb{R}^d \times (0, T) \).

\[\square\]

**Corollary 2.2.** If

\[
\begin{cases}
  u_t - \Delta u + hu \leq 0, & \text{in } \mathbb{R}^d \times (0,T) \\
  \text{for } c > 0, |u| \leq Ke^{c|x|^2} \text{ uniformly in } t \\
  u(x,0) \leq 0
\end{cases}
\]  

(2.3)

where \( \Delta \) is the Laplacian and \( h \) is bounded, then \( u \leq 0 \) in \( \mathbb{R}^d \times (0, T) \).
PROOF. This problem can be written like this

\[
\begin{align*}
&-u_t + \Delta u - hu \geq 0, \quad \text{in } \mathbb{R}^d \times (0,T) \\
&\text{for } c > 0, |u| \leq Ke^{c|x|^2} \text{ uniformly in } t \\
&u(x,0) \leq 0
\end{align*}
\]

(2.4)

and it's just Theorem 2.1 with \( a^{ij} = \delta_{ij} \geq 0 \), where \( \delta_{ij} \) is the Kronecker delta.

\( \square \)

**Corollary 2.3.** If

\[
\begin{align*}
&u_t - u_{xx} + hu \leq 0, \quad \text{in } \mathbb{R} \times (0,T) \\
&\text{for } c > 0, |u| \leq Ke^{c|x|^2} \text{ uniformly in } t \\
&u(x,0) \leq 0
\end{align*}
\]

(2.5)

and \( h \) is bounded, then \( u \leq 0 \) in \( \mathbb{R} \times (0,T) \).

PROOF. This is just corollary 2.2 with dimension \( d = 1 \). \( \square \)
WEIGHTED SPACES

1. Weighted Lebesgue Spaces

Let \( k \in \mathbb{N}, \, 1 \leq p < \infty \) and \( \alpha \in \mathbb{N}^d \) be a multindex of length \( |\alpha| \leq k \) and let \( w \) be a function that is measurable and positive almost everywhere, then \( w \) is called a weight and we define the weighted Lebesgue space as

\[
L^p(\mathbb{R}^d, w) = \left\{ [f] : \int_{\mathbb{R}^d} |f(x)|^p w(x) \, dx < \infty \right\}
\]  

(3.1)

equipped with the norm

\[
\|f\|_{L^p(\mathbb{R}^d, w)} = \left( \int_{\mathbb{R}^d} |f(x)|^p w(x) \, dx \right)^{\frac{1}{p}}
\]  

(3.2)

and when the domain of the function is clear we simply write

\[
\|f\|_{L^p(\mathbb{R}^d, w)} = \|f\|_{0,p,w}
\]  

(3.3)

THEOREM 3.1. The space \( L^p(\mathbb{R}^d, w) \) equipped with the norm \( \|\cdot\|_{L^p(\mathbb{R}^d, w)} \) is a Banach space.

PROOF. Note that the ordinary Lebesgue spaces are denoted by \( L^p(\mathbb{R}^d, \mu) \) where \( \mu \) is the Lebesgue measure and their norm is given by

\[
\|f\|_{L^p(\mathbb{R}^d, \mu)} = \left( \int_{\mathbb{R}^d} |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}}.
\]  

(3.4)

Now we define a new measure by

\[
v(A) = \int_A w(x) \, d\mu(x) = \int_A d\nu(x)
\]  

(3.5)
for $A \subset \mathbb{R}^d$.

Then we have a new space $L^p(\mathbb{R}^d, \nu)$ whose norm is given

$$||f||_{L^p(\mathbb{R}^d, \nu)} = \left( \int_{\mathbb{R}^d} |f(x)|^p \nu(x) \right)^\frac{1}{p} = \left[ \int_{\mathbb{R}^d} |f(x)|^p w(x) dx \right]^\frac{1}{p}$$

and therefore $L^p(\mathbb{R}^d, w)$ is a Banach Space. \hfill \square

Notice that if $w(x) = 1$, then $L^p(\mathbb{R}^d, w) = L^p(\mathbb{R}^d)$ and if $w(x) \leq 1$, then $L^p(\mathbb{R}^d) \subseteq L^p(\mathbb{R}^d, w)$ and also that if $w(x) \geq 1$, then $L^p(\mathbb{R}^d, w) \subseteq L^p(\mathbb{R}^d)$. These can easily be seen because if $f \in L^p(\mathbb{R}^d)$ and if $w(x) \leq 1$, then it follows that $\int_{\mathbb{R}^d} |f(x)|^p w(x) dx \leq \int_{\mathbb{R}^d} |f(x)|^p dx < \infty$, hence $L^p(\mathbb{R}^d) \subseteq L^p(\mathbb{R}^d, w)$, on the other hand if $w(x) \geq 1$ and $f \in L^p(\mathbb{R}^d, w)$ then we have $\int_{\mathbb{R}^d} |f(x)|^p w(x) dx < \infty$

Notice that if $w(x) = 1$, then $L^p(\mathbb{R}^d, w) = L^p(\mathbb{R}^d)$ and if $w(x) \leq 1$, then $L^p(\mathbb{R}^d) \subseteq L^p(\mathbb{R}^d, w)$ and also that if $w(x) \geq 1$, then $L^p(\mathbb{R}^d, w) \subseteq L^p(\mathbb{R}^d)$. These can easily be seen because if $f \in L^p(\mathbb{R}^d)$ and if $w(x) \leq 1$, then it follows that $\int_{\mathbb{R}^d} |f(x)|^p w(x) dx \leq \int_{\mathbb{R}^d} |f(x)|^p dx < \infty$, hence $L^p(\mathbb{R}^d) \subseteq L^p(\mathbb{R}^d, w)$, on the other hand if $w(x) \geq 1$ and $f \in L^p(\mathbb{R}^d, w)$ then we have $\int_{\mathbb{R}^d} |f(x)|^p w(x) dx < \infty$

**Proposition 3.2.** Let $1 \leq p < \infty$. If the weight $w(x)$ is such that $(w(x))^{-\frac{1}{p-1}} \in L^1_{\text{loc}}(\mathbb{R}^d)$ for $p > 1$ and $\|\frac{1}{w}\|_{L^\infty(B)} < \infty$ for $p = 1$ for every ball $B \subseteq \mathbb{R}^d$ then

$$L^p(\mathbb{R}^d, w) \hookrightarrow L^1_{\text{loc}}(\mathbb{R}^d).$$

Where $\hookrightarrow$ denotes continuous embedding. A weight that satisfies this condition is said to be in $B_p(\mathbb{R}^d)$.

**Proof.** Let $u \in L^p(\mathbb{R}^d, w)$, $\int_{\mathbb{R}^d} |u(x)|^p w(x) dx < \infty$. Let $Q \subset \mathbb{R}^d$ be open. Then

$$\int_Q |u(x)| dx = \int_Q |u(x)||w(x)|^\frac{1}{p} [w(x)]^{-\frac{1}{p}} dx.$$  And then we use the fact that

$$u(x)[w(x)]^{\frac{1}{p}} \in L^p(\mathbb{R}^d)$$

and also the fact that

$$[w(x)]^{-\frac{1}{p}} \in L^{p-1}(Q)$$
and then use Hölder inequality:

\[
\int_Q |u(x)| dx \leq \left[ \int_Q |u(x)|^p w(x) dx \right]^{\frac{1}{p}} \left[ \int_Q (w(x))^{-\frac{1}{p'-1}} dx \right]^{\frac{p-1}{p}} \leq C ||u||_{L^p(Q,w)} \leq C ||u||_{L^p(\mathbb{R}^d,w)} < \infty.
\] (3.6)

That was the case when \( p > 1 \), Now consider the case when \( p = 1 \). \( u \in L^1(\mathbb{R}^d, w) \Rightarrow \int_{\mathbb{R}^d} |u(x)|w(x)dx < \infty \), and then we use the fact that \( \frac{1}{w} \in L^\infty(Q) < \infty \). Then

\[
\int_Q |u(x)| dx = \int_Q |u(x)|w(x) [w(x)]^{-1} dx = \int_Q |u(x)|w(x) \frac{1}{w(x)} dx \leq K \int_Q |u(x)|w(x) dx \leq K \int_{\mathbb{R}^d} |u(x)|w(x)dx = K ||u||_{L^1(\mathbb{R}^d,w)} < \infty.
\]

\[\blacksquare\]

This proposition suggests that if the conditions of the proposition are satisfied, then convergence in \( L^p(\mathbb{R}^d,w) \) implies convergence in \( L^1_{loc}(\mathbb{R}^d) \), and because we know that \( L^p(\mathbb{R}^d,w) \subset L^1_{loc}(\mathbb{R}^d) \subset D'(\mathbb{R}^d) \) then any function in \( L^p(\mathbb{R}^d,w) \) has a distributional derivative \( D^\alpha u \). Note that if the conditions of the proposition are not satisfied then the distributional derivative may not exist as the following example shows (see [58]):

1.0.1. Example. Let \( d = 1 \), \( \Omega = (-\frac{1}{2}, \frac{1}{2}) \), \( p > 1 \) and \( w(x) = |x|^{p-1} \). Then we can see that \([w(x)]^{-\frac{1}{p'-1}} \notin L^1_{loc}(\Omega) \) because \([w(x)]^{-\frac{1}{p'-1}} = |x|^{-1} \). So let’s take the function \( u(x) = |x|^{-1} |\log|x||^4 \) where \(-1 < \lambda < \frac{1}{p} \).
Figure 3.1. The weight $w(x)$ for different values of $p > 0$.

Figure 3.2. The weight $w(x)^{-\frac{1}{p-1}}$

\[
\|u\|_{L^p(\Omega, w)}^p = \int_{-\frac{1}{2}}^{1} |u(x)|^p w(x) dx = \int_{-\frac{1}{2}}^{1} |x|^{-p} log|x|^{\lambda p} |x|^{p-1} dx = \int_{-\frac{1}{2}}^{1} |x|^{-1} log|x|^{\lambda p} dx
\]

Let $t = log x, dt = \frac{dx}{x},$

\[
\|u\|_{L^p(\Omega, w)}^p = 2 \int_{0}^{\log 2} |t|^{\lambda p} dt
\]

Let $\tilde{t} = -t, d\tilde{t} = -dt$
\[ ||u||^p_{L^p(\Omega, w)} = -2 \int_0^\infty |\tilde{t}|^\lambda p \, d\tilde{t} = 2 \int_0^\infty t^{\lambda p} \, dt \]
\[ = 2 \frac{t^{\lambda p + 1}}{\lambda p + 1} \bigg|_{\log 2}^\infty < \infty \]

because \( \lambda p + 1 < 0, \lambda p < -1 \). So \( u \in L^p(\Omega, w) \) but on the other hand \( u \notin L^{1}_{loc}(\Omega) \) because since \(-1 < \lambda < -\frac{1}{p}\) we get,
\[ \int_{-\frac{1}{4}}^{\frac{1}{4}} |u(x)| \, dx = \int_{-\frac{1}{4}}^{\frac{1}{4}} |x|^{-1} |log|x||^d \, dx = 2 \int_{0}^{\frac{1}{4}} |x|^{-1} |log|x||^d \, dx = 2 \int_{0}^{\frac{1}{4}} x^{-1} |log|x||^d \, dx \]

let \( t = log x \), then \( dt = \frac{dx}{x} \)
\[ \int_{-\frac{1}{4}}^{\frac{1}{4}} |u(x)| \, dx = 2 \int_{-\infty}^{-\log 4} |t|^d \, dt \]

let \( \tilde{t} = -t, d\tilde{t} = -dt \)
\[ \int_{-\frac{1}{4}}^{\frac{1}{4}} |u(x)| \, dx = -2 \int_{\log 4}^{\infty} |\tilde{t}|^\lambda \, d\tilde{t} = 2 \int_{\log 4}^{\infty} t^{\lambda} \, dt = 2 \frac{t^{\lambda + 1}}{\lambda + 1} \bigg|_{\log 4}^{\infty} = \infty, \]
because \( \lambda + 1 > 0 \) so \( \lambda > -1 \).

A weight \( w \) is said to satisfy an \( A_p \) Muckenhoupt condition or is in \( A_p(\mathbb{R}^d) \) if:

(1) \( w \in L^1_{loc}(\mathbb{R}^d) \)

(2) \( \sup_Q \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q [w(x)]^{-\frac{1}{p-1}} \, dx \right)^{p-1} < \infty \)

where \( 1 < p < \infty \), and the supremum is taken over all cubes \( Q \) in \( \mathbb{R}^d \).

**Lemma 3.3.** A weight \( w \) satisfies the \( A_p \) condition if and only if the Hardy-Littlewood maximal operator is bounded on \( L^p(\mathbb{R}^d, w) \).

**Proof.** The operator \( M : L^p(\mathbb{R}^d, w) \rightarrow L^p(\mathbb{R}^d, w) \) is given by
\[ (Mf)(x) = \sup_Q \frac{1}{|Q|} \int_Q |f(y)| \, dy \]
where the supremum is taken over all cubes \( Q \) containing \( x \). Let

27
\[ \sup_Q \left( \frac{1}{|Q|} \int_Q w(x)dx \right) \left( \frac{1}{|Q|} \int_Q [w(x)]^{-\frac{1}{p-1}} dx \right)^{p-1} = c_{p,w} < \infty. \] (3.7)

And the new measure \( w(E) = \int_E w(x)dx \). It can be shown that this measure is doubling, that is if \( B = B(x,r) \) and \( 2B = B(x,2r) \) then \( w(2B) \leq cw(B) \). These measures are mutually absolutely continuous with respect to the Lebesgue measure. This lemma states that if \( w \in A_p \) then there exists a constant \( c > 0 \) depending only on \( c_{p,w} \) such that (see [8], [7]):

\[ ||Mf||_{L^p(\mathbb{R}^d,w)} \leq c ||f||_{L^p(\mathbb{R}^d,w)} \] (3.8)

whenever \( f \in L^p(\mathbb{R}^d,w) \) and conversely, if this inequality holds for all \( f \in L^p(\mathbb{R}^d,w) \) with \( c \) independent of \( f \), then \( w \in A_p \).

\[ \square \]

**Lemma 3.4.** Suppose \( \eta \in D(\mathbb{R}^d) \) is non-negative with \( \int_{\mathbb{R}^d} \eta(x)dx = 1 \). Suppose furthermore that \( \eta \) is radial and decreasing, that is, \( \eta(x) = \eta(y) \geq \eta(z) \) if \( |x| = |y| \leq |z| \). Let \( f \in L^1_{loc}(\mathbb{R}^d) \) Then

\[ |\eta * f| \leq Mf \] (3.9)

almost everywhere in \( \mathbb{R}^d \) where

\[ (\eta * f)(x) = \int_{\mathbb{R}^d} \eta(x-y)f(y)dy \]

for example

\[ \eta(x) = \begin{cases} a\exp\left(-\frac{1}{|x|^2-1}\right), & \text{if } |x| < 1 \\ 0, & \text{if } |x| \geq 1 \end{cases} \] (3.10)
where the constant $a$ is chosen such that $\int_{\mathbb{R}^d} \eta(x)dx = 1$ then (see [8]) $\eta$ satisfies the conditions of this lemma and also the standard mollifiers satisfy this condition also $\eta_j(x) = j^d \eta(jx)$, for $j = 1,2,...$

**Lemma 3.5.** If $w \in A_p$ and $f \in L^p(\mathbb{R}^d,w)$ then $\eta_j * f \rightharpoonup f$ in $L^p(\mathbb{R}^d,w)$.

**Proof.** First note that the assertion is trivial if $f$ is continuous. Moreover from the previous lemmas we know that there exists a constant $c$ depending on the $A_p$ weight $w$ such that

$$||\eta_j * g||_{L^p(\mathbb{R}^d,w)} \leq ||Mg||_{L^p(\mathbb{R}^d,w)} \leq c||g||_{L^p(\mathbb{R}^d,w)}$$

for all $g \in L^p(\mathbb{R}^d,w)$. To complete the proof, fix $\epsilon > 0$ and let $h$ be a continuous function with $||h - f||_{L^p(\mathbb{R}^d,w)} < \epsilon$. Choosing $j$ such that $||h - \eta_j * h||_{L^p(\mathbb{R}^d,w)} < \epsilon$ we have

$$||f - \eta_j * f||_{L^p(\mathbb{R}^d,w)} \leq ||f - h||_{L^p(\mathbb{R}^d,w)} + ||h - \eta_j * h||_{L^p(\mathbb{R}^d,w)} + ||\eta_j * h - \eta_j * f||_{L^p(\mathbb{R}^d,w)}$$

$$< \epsilon + \epsilon + c||h - f||_{L^p(\mathbb{R}^d,w)} < (c + 2)\epsilon.$$ 

Let $\epsilon \rightarrow 0$ and we get the lemma. \hfill \Box

**Corollary 3.6.** $D(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d,w)$

**Lemma 3.7.** Let $g \in L^p(\mathbb{R}^d,w)$ and let $g_n \in L^p(\mathbb{R}^d,w)$ with $||g||_{L^p(\mathbb{R}^d,w)} \leq c$, $1 < p < \infty$. If $g_n(x) \rightharpoonup g(x)$ a.e in $\mathbb{R}^d$ then $g_n \rightharpoonup g$ in $L^p(\mathbb{R}^d,w)$ where $\rightharpoonup$ denotes weak convergence.

**Proof.** $g_n \in L^p(\mathbb{R}^d,w)$ implies that $g_n \left(\frac{1}{w^{\frac{1}{p}}} \right) \in L^p(\mathbb{R}^d)$ which in turn implies that $g_n(w)^{\frac{1}{p}} \rightharpoonup g(w)^{\frac{1}{p}}$ a.e in $\mathbb{R}^d$ then we have (see [4]) $g_n(w)^{\frac{1}{p}} \rightharpoonup g(w)^{\frac{1}{p}}$ in $L^p(\mathbb{R}^d)$. Moreover for all $\psi \in L^q(\mathbb{R}^d, [w]^{1-q})$ we have $\psi \left(\frac{1}{w^{\frac{1}{q}}} \right) \in L^q(\mathbb{R}^d)$ then

$$\int_{\mathbb{R}^d} g_n \psi dx \rightharpoonup \int_{\mathbb{R}^d} g \psi dx$$

that is

$$g_n \rightharpoonup g$$ in $L^p(\mathbb{R}^d,w)$. 

29
**Lemma 3.8.** The space \( S(\mathbb{R}^d) \) of Schwartz functions is dense in \( L^p(\mathbb{R}^d, w) \) for \( 1 < p < \infty \).

**Proof.** Since we have established that \( D(\mathbb{R}^d) \) is dense in \( L^p(\mathbb{R}^d, w) \) and we know that \( D(\mathbb{R}^d) \subset S(\mathbb{R}^d) \), all we have to do is to prove that \( S(\mathbb{R}^d) \subset L^p(\mathbb{R}^d, w) \) and we will be done. It is clear that for large enough \( k \),

\[
\int_{\mathbb{R}^d} \frac{w(x)}{(1 + |x|)^k} \, dx < \infty
\]

but if \( u \in S(\mathbb{R}^d) \) then \( |u(x)| \leq \frac{C_p}{(1 + |x|)^k} \) for large enough \( k \) (see [9])

\[
\int_{\mathbb{R}^d} |u(x)|^p w(x) \, dx \leq C_k^p \int_{\mathbb{R}^d} \frac{w(x)}{(1 + |x|)^k} \, dx < \infty.
\]

Hence \( S(\mathbb{R}^d) \subset L^p(\mathbb{R}^d, w) \) and also \( D(\mathbb{R}^d) \subset S(\mathbb{R}^d) \subset L^p(\mathbb{R}^d, w) \) and by the fact that \( D(\mathbb{R}^d) \) is dense in \( L^p(\mathbb{R}^d, w) \) it follows that \( S(\mathbb{R}^d) \) is dense in \( L^p(\mathbb{R}^d, w) \). \( \square \)

**Lemma 3.9.** Let \( w \in B_p(\mathbb{R}^d) \), \( \phi \in D(\mathbb{R}^d) \) and \( \alpha \in \mathbb{N}^d \) be a fixed multindex then the function \( T_\alpha : L^p(\mathbb{R}^d, w) \to \mathbb{R} \) defined by \( T_\alpha(u) = \int_{\mathbb{R}^d} u D^\alpha \phi \, dx \) defines a continuous linear functional \( T_\alpha \) on \( L^p(\mathbb{R}^d, w) \)

**Proof.** Let \( Q = \text{supp}(\phi) \), then

\[
|T_\alpha(u)| = \left| \int_{\mathbb{R}^d} u (D^\alpha \phi) \, dx \right| = \left| \int_Q u (D^\alpha \phi) w^{\frac{1}{p}} w^{-\frac{1}{p}} \, dx \right| \leq \int_Q |u| w^{\frac{1}{p}} |D^\alpha \phi| w^{-\frac{1}{p}} \, dx \leq \|u\|_{L^p(\mathbb{R}^d, w)} \|D^\alpha \phi\|_{L^\infty(Q)} \left( \int_Q w^{-\frac{1}{p-1}} \, dx \right)^{\frac{p-1}{p}}
\]

and since \( w \in B_p(\mathbb{R}^d) \) then it follows that \( \int_Q w^{-\frac{1}{p-1}} \, dx < \infty \) and therefore \( |T_\alpha u| < \infty \). \( \square \)

More details on Weighted \( L^p \) Spaces can be found in [31], [30], [34], [35], [45], [46], [54], [58].
2. Weighted Sobolev Spaces

(See [3], [4], [5], [6], [7], [8], [9], [11], [12], [24], [27], [26])

The weighted Sobolev space $W^{k,p}(\mathbb{R}^d, w)$ is defined as the set of all functions $u \in L^p(\mathbb{R}^d, w) \cap L^1_{loc}(\mathbb{R}^d)$ such that their distributional derivative $D^\alpha u$ are again elements of $L^p(\mathbb{R}^d, w) \cap L^1_{loc}(\mathbb{R}^d)$, that is $D^\alpha u$ are regular distributions. The norm in the weighted Sobolev space is given by

$$\|u\|_{W^{k,p}(\mathbb{R}^d, w)} = \left[ \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\mathbb{R}^d, w)}^p \right]^{rac{1}{p}}$$

(3.12)

**Remark 3.10.** If $w \in B_p(\mathbb{R}^d)$, then the assumption $D^\alpha u \in L^p(\mathbb{R}^d, w) \cap L^1_{loc}(\mathbb{R}^d)$ in the definition of the weighted Sobolev space can be replaced by the assumption that $D^\alpha u \in L^p(\mathbb{R}^d, w)$.

**Theorem 3.11.** If $w \in B_p(\mathbb{R}^d)$ then $W^{k,p}(\mathbb{R}^d, w)$ is a Banach space equipped with the norm

$$\|u\|_{W^{k,p}(\mathbb{R}^d, w)} = \left[ \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\mathbb{R}^d, w)}^p \right]^{rac{1}{p}}$$

(3.13)

**Proof.** Let $k = 1$ and let $\{u_n\}_{n=1}^\infty$ be a Cauchy sequence in $W^{1,p}(\mathbb{R}^d, w)$, then $D^\alpha u_n$ is a Cauchy sequence in $L^p(\mathbb{R}^d, w)$ for every $\alpha \in \mathbb{N}^d$, $|\alpha| \leq 1$ and since $L^p(\mathbb{R}^d, w)$ is a Banach space, there exists a function $u_\alpha \in L^p(\mathbb{R}^d, w)$ such that $\lim_{n \to \infty} D^\alpha u_n = u_\alpha$ in $L^p(\mathbb{R}^d, w)$ for every $|\alpha| \leq 1$. Fix $\phi \in D(\mathbb{R}^d)$ and consider the continuous linear functional $T_\alpha$ on $L^p(\mathbb{R}^d, w)$ then $T_\alpha(u_n) \to T_\alpha(u)$ as $n \to \infty$ and also that $T(D^\alpha u_n) \to T(u_\alpha)$ as $n \to \infty$ and since $T_\alpha(u_n) = (-1)^{|\alpha|} T(D^\alpha u_n)$ for every $\phi \in D(\mathbb{R}^d)$. Then $u_\alpha$ is the distributional derivative of $u$ that is $D^\alpha u = u_\alpha$. Since $D^\alpha u \in L^p(\mathbb{R}^d, w) = L^p(\mathbb{R}^d, w) \cap L^1_{loc}(\mathbb{R}^d)$ we have that $u_\alpha \in W^{1,p}(\mathbb{R}^d, w)$ and

$$\|u_n-u\|_{W^{1,p}(\mathbb{R}^d, w)} = \sum_{|\alpha| \leq 1} \|D^\alpha u_n-D^\alpha u\|_{L^p(\mathbb{R}^d, w)} = \sum_{|\alpha| \leq 1} \|D^\alpha u_n-u_\alpha\|_{L^p(\mathbb{R}^d, w)} \to 0 \text{ as } n \to \infty$$

(3.14)
Hence the Cauchy sequence $u_n$ converges to $u$ in $W^{1,p}(\mathbb{R}^d, w)$ that is $W^{1,p}(\mathbb{R}^d, w)$ is complete. The condition $w \in B_p(\mathbb{R}^d)$ is essential as the following example shows (see [58]).

Example
Consider $W^{1,2}((-1,1), w)$ where $w(x) = |x|^\lambda$, $\lambda \in \mathbb{R}$ then since $\frac{1}{w(x)} \notin L^1_{loc}(-1,1)$ we can see that $w \notin B_2(-1,1)$. Let's show that we can choose $\lambda \in \mathbb{R}$ such that $W^{1,2}((-1,1), w)$ is not complete.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.3.png}
\caption{$w(x)$ for different values of $\lambda \geq 1$}
\end{figure}
We show this by constructing a Cauchy sequence \( u_n \subset W^{1,2}((-1,1),w) \) which has no limit in this space. Let's consider the function;
where $\gamma \in \mathbb{R}$.

**Figure 3.6.** The function $u(x)$ for different values of $\lambda \geq 1$

**Figure 3.7.** $u(x)$ for different values of $0 < \lambda \leq 1$
If \( \gamma \leq -1 \), we can see that \( u \notin L^1_{loc}(-1,1) \). Let \( \Omega_1 = (-1,0) \) and \( \Omega_2 = (0,1) \). Obviously \( u \in W^{1,2}(\Omega_1, w) \) and \( \|u\|_{W^{1,2}(\Omega_1, w)} = 0 \). Furthermore \( u \in W^{1,2}(\Omega_2, w) \) under certain condition on \( \gamma \)

\[
\|u\|_{W^{1,2}(\Omega_2, w)}^2 = \int_0^1 x^{2\gamma+\lambda} dx = \frac{x^{2\gamma+\lambda+1}}{2\gamma+\lambda+1} \bigg|_0^1 < \infty
\]

provided \( 2\gamma + \lambda + 1 > 0 \), \( 2\gamma > -\lambda - 1 \), \( \gamma > -\frac{\lambda}{2} - \frac{1}{2} \), \( \gamma > -\frac{\lambda+1}{2} \). For \( 0 < \delta < 1 \) define:

\[
g_\delta(x) = \begin{cases} 
0, & \text{if } -1 < x < \frac{\delta}{2} \\
\frac{2}{\delta}(x-1), & \text{if } \frac{\delta}{2} < x < \delta \\
1, & \text{if } \delta < x < 1
\end{cases}
\]

(3.16)
And \( v_\delta(x) = u(x)g_\delta(x) \), then so long as \( \gamma > -\frac{1}{2} - \frac{1}{2} \) we can see that

\[
\lim_{\delta \to 0} \| u - v_\delta \|_{W^{1,2}(\Omega_i,w)} = 0, \text{ for } i = 1, 2
\]  

(3.17)

If we denote \( u_n = v_{\frac{1}{n}} \), then evidently \( u_n \in W^{1,2}(\Omega, w) \) and so \( u_n \) is a cauchy sequence in \( W^{1,2}(\Omega, w) \) for both \( i = 1, 2 \), but then \( u_n \) is a Cauchy sequence in \( W^{1,2}(\Omega, w) \) too since

\[
\| u_n \|_{W^{1,2}(\Omega, w)}^2 = \| u_n \|_{W^{1,2}(\Omega_1,w)}^2 + \| u_n \|_{W^{1,2}(\Omega_2,w)}^2.
\]  

(3.18)
Let us suppose that $W^{1,2}(\Omega, w)$ is complete, then there exists an element $u^* \in W^{1,2}(\Omega, w)$ such that

$$\lim_{n \to \infty} \|u^* - u_n\|_{W^{1,2}(\Omega, w)} = 0. \quad (3.19)$$

Note that

$$\lim_{n \to \infty} \|u^* - u_n\|_{W^{1,2}(\Omega_i, w)} = 0, \text{ for } i = 1, 2. \quad (3.20)$$

And therefore $u = u^*$ a.e in $\Omega_i$ for $i = 1, 2$ that is $u = u^*$ a.e in $\Omega$. The function $u^* \in W^{1,2}(\Omega, w)$ and therefore $u^* \in L^1_{loc}(\Omega)$, Hence also $u \in L^1_{loc}(\Omega)$ and this is a contradiction. For example when $\lambda = 2$ and $-\frac{3}{2} < \gamma < -1$. So $W^{1,2}((-1,1), w)$ is not complete.

Remark 3.12. In the previous example, for a given function $u \in W^{1,2}((-1,1), w)$ we have constructed a Cauchy sequence $u_n \subset W^{1,2}((-1,1), w)$ which approximates $u$ in both $W^{1,2}(\Omega_i, w)$ for $i = 1, 2$. We can choose another sequence $u^*_n \subset C^\infty([-1,1])$ but still with the same properties. This can be done by using the imbedding

$$W^{1,2}(-1,1) \subset W^{1,2}((-1,1), w)$$

which holds for $\lambda \geq 0$ and the facts that the above functions $u_n \in W^{1,2}(-1,1)$ and $C^\infty([-1,1])$ is dense in $W^{1,2}(-1,1)$

The Space $W^{1,p}_0(\Omega, w)$

The classical Sobolev space $W^{1,p}(\Omega)$ if often defined (for a reasonable domain $\Omega$) as the closure of the set $C^\infty(\bar{\Omega})$ in the corresponding norm $\|\cdot\|_{W^{1,p}(\Omega)}$. If we want to proceed analogously in the case of weighted spaces, we need first to establish that $C^\infty(\bar{\Omega}) \subset W^{1,p}(\Omega, w)$. This requirement would however exclude a great number of weights for example weights of the type $|x-x_0|^{-\lambda}$ for large $\lambda > 0$, $x_0 \in \bar{\Omega}$ fixed while it is clear that such weights are in $B_p(\Omega)$. And also even if the condition is fulfilled,
then the completion could lead to Banach Spaces with elements which are non-regular distributions.

In various applications, particularly for the investigation of the Dirichlet problem for elliptic partial differential equations, we need the space \( W^{1,p}_0(\Omega, w) \) defined as the closure of \( C^\infty_0(\Omega) \) with respect to the norm

\[
\|u\|_{W^{1,p}(\Omega, w)} = \left[ \|u\|_{L^p(\Omega, w)}^p + \|Du\|_{L^p(\Omega, w)}^p \right]^\frac{1}{p}.
\]

(3.21)

In order to introduce this space we need the inclusion \( C^\infty_0(\Omega) \subset W^{1,p}(\Omega, w) \) which is evidently satisfied if \( w \in L^1_{loc}(\Omega) \). So we let \( w \in B_p(\Omega) \cap L^1_{loc}(\Omega) \) and we define \( W^{1,p}_0(\Omega, w) = \overline{C^\infty_0(\Omega)} \) where the closure is taken with respect to the norm

\[
\|u\|_{W^{1,p}(\Omega, w)} = \left[ \|u\|_{L^p(\Omega, w)}^p + \|Du\|_{L^p(\Omega, w)}^p \right]^\frac{1}{p}.
\]

**Lemma 3.13.** \( C^\infty_0(\Omega) \subset W^{1,p}(\Omega, w) \) if and only if \( w \in L^1_{loc}(\Omega) \)

**Proof.** Assume \( w \in L^1_{loc}(\Omega) \) and \( u \in C^\infty_0(\Omega) \) then

\[
\|u\|_{W^{1,p}(\Omega, w)} = \left[ \|u\|_{L^p(\Omega, w)}^p + \|Du\|_{L^p(\Omega, w)}^p \right]^\frac{1}{p} = \left[ \int_\Omega |u(x)|^p w(x)dx + \int_\Omega |Du(x)|^p w(x)dx \right]^\frac{1}{p}
\]

\[
\leq \left[ \|u^p\|_{L^\infty(Q)} \int_Q w(x)dx + \|(Du)^p\|_{L^\infty(Q)} \int_Q w(x)dx \right]^\frac{1}{p}
\]

\[
= \left[ \left( \|u^p\|_{L^\infty(Q)} + \|(Du)^p\|_{L^\infty(Q)} \right) \int_Q w(x)dx \right]^\frac{1}{p}
\]

\[
= \left( \|u^p\|_{L^\infty(Q)} + \|(Du)^p\|_{L^\infty(Q)} \right)^\frac{1}{p} \left( \int_Q w(x)dx \right)^\frac{1}{p} < \infty
\]

so \( u \in W^{1,p}(\Omega, w) \) and therefore \( C^\infty_0(\Omega) \subset W^{1,p}(\Omega, w) \).

Conversely assume \( C^\infty_0(\Omega) \subset W^{1,p}(\Omega, w) \) and let \( Q \subset \Omega \) be a compact subset of \( \Omega \) then there exists a function \( \phi \in C^\infty_0(\Omega) \) such that \( D^\alpha \phi(x) = 1 \) in \( Q \). (see [58], [8]) Then

\[
0 \leq \int_Q w(x)dx = \int_Q |D^\alpha \phi(x)|^p w(x)dx \leq \int_\Omega |D^\alpha \phi(x)|^p w(x)dx = \|D^\alpha \phi\|_{L^p(\Omega, w)}^p
\]

\[
\leq \|\phi\|_{L^p(\Omega, w)}^p + \|D^\alpha \phi\|_{L^p(\Omega, w)}^p = \|\phi\|_{W^{1,p}(\Omega, w)}^p < \infty
\]

38
So $w \in L_{loc}^1(\Omega)$.

So this lemma shows that for some weights the inclusion $C_0^\infty(\Omega) \subset W^{1,p}(\Omega,w)$ might not hold and we should therefore devise an alternative strategy. We define an exceptional set:

$$M_0(w) = \left\{ x \in \Omega : \int_{\Omega \cap B(x)} w(t) dt = \infty \text{ for every neighbourhood of } x, B(x) \right\} \quad (3.22)$$

Then if $w \in L_{loc}^1(\Omega)$ then $M_0(w) = \emptyset$

**Lemma 3.14.** Assume $w \notin L_{loc}^1(\Omega)$ then

1. $M_0(w)$ is a non-empty closed subset of $\Omega$
2. $w \in L_{loc}^1(\Omega - M_0(w))$
3. If $w$ is continuous a.e in $\Omega$ then $|M_0(w)| = 0$.

**Definition 3.15.** If $w \notin L_{loc}^1(\Omega)$ then define

$$M_p(w) = \left\{ x \in \Omega : \int_{\Omega \cap B(x)} [w(t)]^{-\frac{1}{p-1}} dt = \infty \text{ for every neighbourhood of } x, B(x) \right\} \quad (3.23)$$

if $w \in B_p(\Omega)$ then $M_p = \emptyset$. define $W_0^{1,p}(\Omega,w) = \overline{V}$ where

$$V = \left\{ f : f = g|_{\Omega - M_p(w)}, g \in C_0^\infty(\Omega - M_0(w)) \right\}, \quad (3.24)$$

where the closure is taken with respect to the norm,

$$\|u\|_{W^{1,p}(\Omega,w)} = \left( \|u\|_{L^p(\Omega,w)}^p + \|Du\|_{L^p(\Omega,w)}^p \right)^{\frac{1}{p}}. \quad (3.25)$$

So we have obtained a Banach space $W_0^{1,p}(\Omega,w)$ which is a subset of $W^{1,p}(\Omega,w)$

**Remark 3.16.** If $w \in B_p(\Omega)$ then $M_p(w) = \emptyset$ and so $W_0^{1,p}(\Omega,w) = C_0^\infty(\Omega - M_0(w))$.

If in addition we have that $w \in L_{loc}^1(\Omega)$ then $M_0(w) = \emptyset$ and so the definition goes back to $W_0^{1,p}(\Omega,w) = \overline{C_0^\infty(\Omega)}$. Note that $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega,w)$ and the
space $\left(W^{1,p}_0(\Omega,w), \|\cdot\|_{W^{1,p}(\Omega,w)}\right)$ is a reflexive Banach space and we recall the dual of $W^{1,p}_0(\Omega,w)$ is equivalent to $W^{-1,p'}(\Omega,w^*)$ where $w^* = 1 - p'$ and $p' = \frac{p}{p-1}$. You notice that (see \[4, 7\])

$$\left[ W^{1,p}_0(\Omega,w) \right]^* = W^{-1,p'}(\Omega,w^*)$$

(3.26)

and therefore it turns out that

$$\left[ W^{-1,p'}(\Omega,w^*) \right]^* = W^{1,p''}(\Omega,w^{**})$$

(3.27)

where $w^{**} = (w^*)^{1-p''}$ and $p'' = \frac{p'}{p-1}$ since $\frac{1}{p} + \frac{1}{p'} = 1 \Rightarrow \frac{p'}{p-1} = p - 1$. Since $p'' = \frac{p}{p-1} = \left(\frac{p}{p-1}\right)\left(\frac{p-1}{1}\right) = p$ and so $1 - p'' = 1 - p$

$$w^{**} = (w^*)^{1-p} = \left(w^{1-p}\right)^{1-p} = \left(w^{1-p}\right)^{p-1} = w$$

so

$$\left[ W^{1,p}_0(\Omega,w) \right]^{**} = \left[ W^{-1,p'}(\Omega,w^*) \right]^* = W^{1,p''}(\Omega,w^{**}) = W^{1,p}_0(\Omega,w)$$

(3.28)

The expression $\|u\|^* = \|Du\|_{L^p(\Omega,w)}$ is a norm defined on $W^{1,p}_0(\Omega,w)$ and it is equivalent to the norm $\|u\|_{W^{1,p}(\Omega,w)} = \left(\|u\|_{L^p(\Omega,w)}^p + \|Du\|_{L^p(\Omega,w)}^p\right)^{\frac{1}{p}}$. Then $\left(W^{1,p}_0(\Omega,w), \|\cdot\|^*\right)$ is a uniformly convex (and therefore reflexive) Banach space. We assume there exists a weight function $\sigma$ on $\Omega$ and a parameter $q$, $1 < q < \infty$, such that $\sigma^{1-q'} \in L^1(\Omega)$ with $q' = \frac{q}{q-1}$ and such that the Hardy Inequality is satisfied:

$$\|u\|_{L^q(\Omega,\sigma)} \leq C\|Du\|_{L^p(\Omega,w)}$$

(3.29)

for every $u \in W^{1,p}_0(\Omega,w)$

with the constant $C$ independent of $u$. Moreover the imbedding $W^{1,p}_0(\Omega,w) \hookrightarrow L^q(\Omega,\sigma)$ determined by the above inequality is compact.
LEMMA 3.17. Assume the assumption above is satisfied. Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let $u \in W^{1,p}_0(\Omega, w)$, then $F(u) \in W^{1,p}_0(\Omega, w)$. Moreover if the set $D$ of discontinuity points of $F'$ is finite, then

$$
\frac{\partial}{\partial x_i}(F \circ u) = \begin{cases} 
F'(u) \frac{\partial u}{\partial x_i} & \text{a.e in } \{ x \in \Omega : u(x) \notin D \} \\
0 & \text{a.e in } \{ x \in \Omega : u(x) \in D \}
\end{cases}
$$

(3.30)

PROOF. Consider first the case $F \in C^1(\mathbb{R})$ and $F' \in L^{\infty}(\mathbb{R})$. Let $u \in W^{1,p}_0(\Omega, w)$. Since $D(\Omega)$ is dense in $W^{1,p}_0(\Omega, w)$, there exists a sequence $\{u_n\} \subset D(\Omega)$ such that

$$
u_n \to u \quad \text{in} \quad W^{1,p}_0(\Omega, w),$$

passing to a subsequence, we can assume that

$$
u_n \to \nu \quad \text{a.e in} \quad \Omega$$

$$
\nabla \nu_n \to \nabla \nu \quad \text{a.e in} \quad \Omega.
$$

Then $F(\nu_n) \to F(\nu)$ a.e in $\Omega$. On the other hand, from the relation

$$|F(\nu_n)|^p w \leq \|F'\|_{L^{\infty}(\Omega)} |u_n|^p w \quad \text{(3.31)}$$

and

$$
\left| \frac{\partial F(\nu_n)}{\partial x_i} \right|^p w = \left| F'(\nu_n) \frac{\partial u_n}{\partial x_i} \right|^p w \leq M \left| \frac{\partial u_n}{\partial x_i} \right|^p w
$$

(3.32)

we deduce that the function $F(\nu_n)$ remains bounded in $W^{1,p}_0(\Omega, w)$. Thus going to a further subsequence, we obtain $F(\nu_n) \to v$ in $W^{1,p}_0(\Omega, w)$ then we conclude that $v = F(\nu) \in W^{1,p}_0(\Omega, w)$. We now turn to the general case. Taking convolutions with mollifiers $\rho_n$ in $\mathbb{R}$, we have $F_n = F \ast \rho_n$, $F_n \in C^1(\mathbb{R})$ and $F'_n \in L^{\infty}(\mathbb{R})$. Then by the first case we have $F(u_n) \in W^{1,p}_0(\Omega, w)$. Since $F_n \to F$ uniformly in every compact subset, we have $F_n(u) \to F(u)$ a.e in $\Omega$. On the other hand, $\{F_n(u)\}$ is bounded in $W^{1,p}_0(\Omega, w)$. Then for a subsequence $F_n(u) \to \bar{v}$ in $W^{1,p}_0(\Omega, w)$ a.e in $\Omega$, then $\bar{v} = F(u) \in W^{1,p}_0(\Omega, w).$
Lemma 3.18. Assume that our assumption holds. Let \( u \in W_0^{1,p}(\Omega, w) \), then \( u^+ = \max\{u, 0\} \) and \( u^- = \max\{-u, 0\} \) are in \( W_0^{1,p}(\Omega, w) \). Moreover we have

\[
\frac{\partial u^+}{\partial x_i} = \begin{cases} \frac{\partial u}{\partial x_i}, & u > 0 \\ 0, & u \leq 0 \end{cases} \quad (3.33)
\]

\[
\frac{\partial u^-}{\partial x_i} = \begin{cases} 0, & u \geq 0 \\ -\frac{\partial u}{\partial x_i}, & u < 0 \end{cases} \quad (3.34)
\]

Lemma 3.19. Assume our assumption holds. Let \( \{u_n\} \) be a sequence of \( W_0^{1,p}(\Omega, w) \) such that \( u_n \rightharpoonup u \) weakly in \( W_0^{1,p}(\Omega, w) \). Then \( u_n^+ \rightharpoonup u^+ \) weakly in \( W_0^{1,p}(\Omega, w) \) and \( u_n^- \rightharpoonup u^- \) weakly in \( W_0^{1,p}(\Omega, w) \).

Proof. Since \( u_n \rightharpoonup u \) weakly in \( W_0^{1,p}(\Omega, w) \), then by the previous proof we have for a subsequence \( u_n \rightharpoonup u \) in \( L^q(\Omega, \sigma) \) and a.e in \( \Omega \). On the other hand

\[
\|u_n\| = \|Du_n\|_{L^p(\Omega, w)} \geq \|Du_n\|_{L^p(u_n \geq 0, \omega)} \geq \|Du_n^+\|_{L^p(\Omega, w)} = \|u_n^+\|.
\]

Then \( \{u_n^+\} \) is bounded in \( W_0^{1,p}(\Omega, w) \) hence by the previous proof \( u_n^+ \rightharpoonup u^+ \) weakly in \( W_0^{1,p}(\Omega, w) \). A similar argument is used to prove that \( u_n^- \rightharpoonup u^- \) weakly in \( W_0^{1,p}(\Omega, w) \).

Theorem 3.20. For the weight \( w \) we can define a new Borel measure by \( \nu(E) = \int_E w(x) dx \). Let \( w(x) \) be a continuous, bounded weight. For every \( r > 0 \) let \( \Omega_r, S_r \) be defined by

\[
\Omega_r = \{x \in \Omega : |x| > r\}
\]

\[
S_r = \{x \in \Omega : |x| = r\}
\]
Moreover, let us denote by $A_r$ the surface area, with respect to the weight $w$ of $S_r$. Then if the imbedding $W^{1,p}(\Omega, w) \hookrightarrow L^p(\Omega, w)$ is compact:

1. For every $\epsilon > 0$, $\delta > 0$ there exists $R > 0$ such that if $r \geq R$,

$$\nu(\Omega_r) \leq \delta \nu(\{x \in \Omega: r - \epsilon \leq |x| \leq r\});$$

2. If $A_r$ is positive and ultimately non-increasing as $r \to +\infty$ then for every $\epsilon > 0$

$$\lim_{r \to +\infty} \frac{A_{r+\epsilon}}{A_r} = 0 \quad (3.36)$$

Proof. The Thesis follows an easy extension of the argument in [3] and [31], where the lebesgue measure $\mu$ is replaced with $\nu$, $\lambda$-fat cubes are replaced by $(\lambda, w)$-fat cubes and $A_r$ is computed with respect to the weight $w$. \hfill \Box

Corollary 3.21. Let $w(x)$ be a continous upper bounded weight. If the imbedding $W^{1,p}(\Omega, w) \hookrightarrow L^p(\Omega, w)$ is compact then for every $k \in \mathbb{Z}$ $\lim_{r \to +\infty} e^{kr} \nu(\Omega_r) = 0$

Definition 3.22. Let $w$ be a lower semicontinuos weight defined on an open set $\Omega \subset \mathbb{R}^d$. Let us suppose that $w$ vanishes only on a closed subset $\Omega_0 \subset \Omega$. Moreover, let

$$\Omega_\infty := \{x \in \Omega: w(x) = +\infty\}. \quad (3.37)$$

And suppose that $\Omega_\infty$ is closed. Both $\Omega_0$ and $\Omega_\infty$ have Lebesgue measure equal to zero. Moreover, we suppose that $w$ is bounded from above and from below by positive constants on any compact set $K \subset \Omega \setminus (\Omega_0 \cup \Omega_\infty)$. Let us denote by $\Omega_w$ the subgraph of the weight $w(x)$, that is, the open set:

$$\Omega_w = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}: x \in \Omega, 0 < y < w(x)\}. \quad (3.38)$$

And consider the map $J : W^{1,p}(\Omega, w) \to W^{1,p}(\Omega_w)$ defined by $(Ju)(x, y) = u(x)$ a.e. $J$ is well defined. Its easy to see that if $u \in J(W^{1,p}(\Omega, w))$ then the distributional derivative in the y-direction $\nabla_y u$ is equal to zero. $J$ is an isometry of $W^{1,p}(\Omega, w)$ onto $J(W^{1,p}(\Omega, w))$ since for every $u \in W^{1,p}(\Omega, w)$
\[
\|Ju\|_{W^{1,p}(\Omega_w)}^p = \int_{\Omega_w} |Ju(x,y)|^p \, dx \, dy + \int_{\Omega_w} |\nabla_x Ju(x,y)|^p \, dx \, dy
\]
\[
= \int_{\Omega} \left[ \int_0^{w(x)} |u(x)|^p \, dy \right] \, dx + \int_{\Omega} \left[ \int_0^{w(x)} |\nabla_x u(x)|^p \, dy \right] \, dx
\]
\[
= \int_{\Omega} |u(x)|^p w(x) \, dx + \int_{\Omega} |\nabla_x u(x)|^p w(x) \, dx.
\]

We denote by \(W^{1,p}_y(\Omega_w)\) the set \(J(W^{1,p}(\Omega_w))\). Moreover we will denote by \(L^p_y(\Omega_w)\) the completion of \(W^{1,p}_y(\Omega_w)\) with respect to the norm of \(L^p(\Omega_w)\).

**Lemma 3.23.** If \(w(x)\) satisfies the conditions above, then \(C_0^\infty(\Omega \setminus (\Omega_0 \cup \Omega_\infty))\) is dense in \(L^p(\Omega, w)\) for \(1 \leq p < +\infty\).

**Proof.** It suffices to show that, given \(f \in L^p(\Omega, w)\), for every \(\varepsilon > 0\) there exists \(g \in D(\Omega \setminus (\Omega_0 \cup \Omega_\infty))\) such that \(\|f - g\|_{L^p(\Omega, w)} < \varepsilon\). Let \(\{\Omega_n\}, \quad n \in \mathbb{N},\) be an exhaustion of \(\Omega\) defined by

\[
\Omega_n = \left\{ x \in \Omega : \min\{d(x, \Omega_0), d(x, \Omega_\infty), d(x, \delta \Omega)\} > \frac{1}{n} \right\}
\]

(3.39)

Let \(f \in L^p(\Omega, w)\); for every \(\varepsilon > 0\) there exists \(\tilde{n}\) such that

\[
\left( \int_{\Omega \setminus \Omega_{\tilde{n}}} |f(x)|^p w(x) \, dx \right)^{\frac{1}{p}} < \frac{\varepsilon}{2}
\]

(3.40)

Since \(w\) is bounded from above and from below by positive constants on \(\Omega_{\tilde{n}}\), \(u \in L^p(\Omega_{\tilde{n}}, w)\) if and only if \(u \in L^p(\Omega_{\tilde{n}})\) and there exists \(C_1, C_2 > 0\) such that for every \(u \in L^p(\Omega_{\tilde{n}}, w)\)

\[
C_2 \|u\|_{L^p(\Omega_{\tilde{n}})} \leq \|u\|_{L^p(\Omega_{\tilde{n}}, w)} \leq C_1 \|u\|_{L^p(\Omega_{\tilde{n}})}
\]

Hence \(f|_{\Omega_{\tilde{n}}} \in L^p(\Omega_{\tilde{n}})\). As a consequence, there exists a function

\[
g \in D(\Omega_{\tilde{n}}) \subset D(\Omega \setminus (\Omega_0 \cup \Omega_\infty))
\]

44
such that
\[ \|g - f\|_{L^p(\bar{\Omega})} < (2C_1)^{-1} \epsilon \quad \text{hence} \quad \|g - f\|_{L^p(\Omega, w)} < \frac{\epsilon}{2}. \]

This implies that
\[ \|g - f\|_{L^p(\Omega, w)} = \|g - f\|_{L^p(\Omega, w)} + \|f\|_{L^p(\Omega \setminus \bar{\Omega})} < \epsilon. \]

Hence, since \( D(\Omega \setminus (\Omega_0 \cup \Omega_\infty)) \subset W^{1,p}(\Omega, w) \), \( W^{1,p}(\Omega, w) \) is dense in \( L^p(\Omega, w) \). Since for every \( u \in W^{1,p}(\Omega, w) \)
\[
\int_{\Omega} |u(x)|^p w(x) dx = \int_{\Omega_w} |J(x, y)|^p dxdy. \tag{3.41}
\]

This leads to the following

\[ \square \]

**Theorem 3.24.** Let \( w(x) \) satisfy the above conditions, then \( J \) can be extended to an isometry

\[ \tilde{J} : L^p(\Omega, w) \rightarrow L^p_y(\Omega_w). \tag{3.42} \]

And as a consequence we have the following, If the subgraph \( \Omega_w \) of \( w(x) \) is such that the embedding

\[ I_{\Omega_w} : W^{1,p}(\Omega_w) \rightarrow L^p(\Omega_w) \tag{3.43} \]

is compact, then the embedding

\[ W^{1,p}(\Omega, w) \rightarrow L^p(\Omega, w) \tag{3.44} \]

is compact.

**Proof.** The embedding \( W^{1,p}(\Omega, w) \rightarrow L^p(\Omega, w) \) is compact if and only if the embedding
is compact but, \( I_y \) coincides with \( P_y \circ I_{\Omega_w} \circ I \), where \( I \) is the immersion

\[
I : W^{1,p}_y(\Omega_w) \longrightarrow W^{1,p}_y(\Omega_w)
\]  \hspace{1cm} (3.46)

and \( P_y \) denotes the projection

\[
P_y : L^p(\Omega_w) \longrightarrow L^p_y(\Omega_w)
\]  \hspace{1cm} (3.47)

defined for a.e \( x \in \Omega \) by

\[
(P_y)(x) = \frac{1}{w(x)} \int_0^{w(x)} u(x,y) dy.
\]  \hspace{1cm} (3.48)

Since \( P_y \) and \( I \) are continuous, then the thesis follows.

**Proposition 3.25.** Let \( \Phi \) be a weight for which the embedding

\[
W^{1,p}(\Omega, \Phi) \hookrightarrow L^p(\Omega, \Phi)
\]  \hspace{1cm} (3.49)

is compact. Let \( w(x) \) be a weight such that there exists \( \alpha, \beta > 0 \) such that a.e in \( \Omega \)

\[
\alpha \Phi(x) \leq w(x) \leq \beta \Phi(x)
\]  \hspace{1cm} (3.50)

then the embedding

\[
W^{1,p}(\Omega, w) \hookrightarrow L^p(\Omega, w)
\]  \hspace{1cm} (3.51)

is compact

**Proof.** It is immediate that \( u \in L^p(\Omega, \Phi) \) if and only \( u \in L^p(\Omega, w) \) and

\[
\alpha \|u\|_{L^p(\Omega, \Phi)} \leq \|u\|_{L^p(\Omega, w)} \leq \beta \|u\|_{L^p(\Omega, \Phi)}.
\]  \hspace{1cm} (3.52)
Analogously $u \in W^{1,p}(\Omega, \Phi)$ if and only if $u \in W^{1,p}(\Omega, w)$ and

$$\alpha \|u\|_{W^{1,p}(\Omega, \Phi)} \leq \|u\|_{W^{1,p}(\Omega, w)} \leq \beta \|u\|_{W^{1,p}(\Omega, \Phi)}. \quad (3.53)$$

Hence if $\{u_n\}$ is a bounded sequence in $W^{1,p}(\Omega, w)$, it is also bounded in $W^{1,p}(\Omega, \Phi)$. Due to the compactness of the embedding $W^{1,p}(\Omega, \Phi) \hookrightarrow L^p(\Omega, \Phi)$ there exists a subsequence $\{u_{n_k}\}$ such that $u_{n_k}$ converges in $L^p(\Omega, \Phi)$ hence $u_{n_k}$ also converges in $L^p(\Omega, w)$. □

**Theorem 3.26.** Let $\Omega$ be an open set in $\mathbb{R}^d$. If

1. there exists a sequence $\{\Omega_N^*\}_{N=1}^{\infty}$ of open subsets of $\Omega$ such that $\Omega_N^* \subseteq \Omega_{N+1}^*$ and for every $N$, the embedding

$$W^{1,p}(\Omega_N^*) \hookrightarrow L^p(\Omega_N^*) \quad (3.54)$$

is compact;

2. there exists a flow $\Phi : U \longrightarrow \Omega$ and a constant $c > 0$ such that if $\Omega_N = \Omega \setminus \Omega_N^*$ then

   - $\Omega_N \times [0,c] \subset U$ for every $N$;
   - $\Phi_t$ is one-to-one for every $t$;
   - there exists $M > 0$ such that for every $(x,t) \in U$ $|\partial_t \Phi(x,t)| \leq M$;

3. The functionals $d_N(t) = \sup_{x \in \Omega_N} |\det J\Phi_t(x)|^{-1}$ satisfy

   - $\lim_{N \to \infty} d_N(c) = 0$;
   - $\lim_{N \to \infty} \int_0^c d_N(t)dt = 0$,

Then the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact.

**Proof.** Recall that a flow on $\Omega$ is a continuously differentiable map $\Phi : U \longrightarrow \mathbb{R}^d$ where $U$ is an open set in $\Omega \times \mathbb{R}$ containing $\Omega \times \{0\}$ with $\Phi(x,0) = x$ for every $x$ in $\Omega$. Moreover, we denote by $\Phi_t$ the map $\Phi_t : x \longrightarrow \Phi(x,t)$, and by $J\Phi_t$ the Jacobian matrix of $\Phi$. The proof of this theorem is in [31]. □
Lemma 3.27. Let \( \Omega = \mathbb{R}^d \), \( w(x) \) be a radial function \( w(x) = g(r) \) where \( r = |x| \) and \( g \in C^1([0, +\infty)) \) is positive, nonincreasing, with bounded derivative \( g' \); then the embedding

\[
W^{1,p}(\Omega, w) \hookrightarrow L^p(\Omega, w)
\]

is compact if and only if

\[
\lim_{s \to 0} \frac{g(s+\epsilon)}{g(s)} = 0 \quad \text{for every} \quad \epsilon > 0.
\]

Proof. Suppose, first, that \( \lim_{s \to 0} \frac{g(s+\epsilon)}{g(s)} = 0 \) for every \( \epsilon > 0 \). Let us consider, on \( \mathbb{R}^d \), polar coordinates \((r, \theta)\). Then the subgraph of \( w \) can be described by

\[
\Omega_w := \{(r, \theta, y) | 0 < y < g(r)\}.
\] (3.55)

For every \( N \in \mathbb{N} \), let us consider the set

\[
(\Omega_w)_N := \{(r, \theta, y) \in \Omega_w | r \geq N\}.
\] (3.56)

Then \((\Omega_w)_N^* := \Omega_w \setminus (\Omega_w)_N\) is bounded and has the cone property; hence, the embedding

\[
W^{1,p}((\Omega_w)_N^*) \hookrightarrow L^p((\Omega_w)_N^*)
\] (3.57)

is compact for every \( N \in \mathbb{N} \). Moreover \((\Omega_w)_N^* \subset (\Omega_w)_{N+1}^* \) for every \( N \in \mathbb{N} \). An easy computation shows that the flow

\[
\Phi(r, \theta, y, t) := \left(r-t, \theta, \frac{g(r-t)}{g(r)} y\right)
\] (3.58)

defined on the set

\[
U := \{(r, \theta, y, t) | 0 < t < r\},
\] (3.59)
satisfies the conditions of theorem 3.26 with $c = 1$. As a consequence, the embedding $I_{\Omega_w}$ is compact, and theorem 3.24 yields the thesis. Conversely, suppose that the embedding $W^{1,p}(\Omega, w) \hookrightarrow L^p(\Omega, w)$ is compact then by theorem 3.20

$$A_r = \int_{|x|=r} g(r)r^{n-1}dr d\theta = C(n)r^{n-1}g(r)$$

must fulfill the condition

$$\lim_{r \to +\infty} \frac{A_{r+\epsilon}}{A_r} = 0.$$  

As a consequence,

$$\lim_{r \to +\infty} \frac{g(r+\epsilon)}{g(r)} = \lim_{r \to +\infty} \frac{C(n)(r+\epsilon)^{n-1}g(r+\epsilon)}{C(n)r^{n-1}g(r)} = 0$$  \hspace{1cm} (3.60)

\[ \square \]

**Remark 3.28.** In particular, for $g(r) = e^{ar}$ and for $g(r) \sim r^\alpha$ as $r \to +\infty$, $\alpha < 0$ we get that there is no compactness.

### 3. Special Weights

Let’s consider the set $\mathbb{R}^d \times (0, T)$ and then we use the special weight,

$$w_\lambda(x) = \exp \left[ \lambda \sqrt{1+|x|^2} \right], \quad \text{where} \quad \lambda \in \mathbb{R},$$

and define the weighted $L^2$ space by :

$$L^2_\lambda(\mathbb{R}^d \times (0, T)) := \left\{ [f] : \int_{\mathbb{R}^d \times (0, T)} |f(x,t)|^2 w_\lambda(x) dx dt < \infty \right\}$$

with the norm :

$$\|f\|_{L^2_\lambda(\mathbb{R}^d \times (0, T))} := \left[ \int_{\mathbb{R}^d \times (0, T)} |f(x,t)|^2 w_\lambda(x) dx dt \right]^{\frac{1}{2}}.$$
By theorem 3.1, the space \( L^2_\lambda(\mathbb{R}^d \times (0, T)) \) is a reflexive Banach space. It is actually a Hilbert space with the inner product given by:

\[
\langle u, v \rangle_{L^2_\lambda(\mathbb{R}^d \times (0, T))} = \langle u \sqrt{w_\lambda}, v \sqrt{w_\lambda} \rangle_{L^2(\mathbb{R}^d \times (0, T))} = \int_{\mathbb{R}^d \times (0, T)} u \sqrt{w_\lambda} v \sqrt{w_\lambda} \, dx \, dt
\]

\[
= \int_{\mathbb{R}^d \times (0, T)} u v \sqrt{w_\lambda} \, dx \, dt \leq \left( \int_{\mathbb{R}^d \times (0, T)} u^2 \sqrt{w_\lambda} \, dx \, dt \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^d \times (0, T)} v^2 \sqrt{w_\lambda} \, dx \, dt \right)^{\frac{1}{2}}
\]

\[
= \| u \|_{L^2_\lambda(\mathbb{R}^d \times (0, T))} \| v \|_{L^2_\lambda(\mathbb{R}^d \times (0, T))} = \| u \sqrt{w_\lambda} \|_{L^2(\mathbb{R}^d \times (0, T))} \| v \sqrt{w_\lambda} \|_{L^2(\mathbb{R}^d \times (0, T))} < \infty
\]

Notice also that the weight \( w_\lambda \in B_2(\mathbb{R}^d \times (0, T)) \) and therefore the inclusion \( C^\infty_0(\mathbb{R}^d \times (0, T)) \subset L^2_\lambda(\mathbb{R}^d \times (0, T)) \) holds.

The weight also satisfies the \( A_2(\mathbb{R}^d \times (0, T)) \) condition which is:

\[
\sup_Q \left( \frac{1}{|Q|} \int_Q w_\lambda(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q \frac{1}{w_\lambda(x)} \, dx \right) < \infty
\]

and also

\[ w \in L^1_{\text{loc}}(\mathbb{R}^d \times (0, T)) \]

and therefore the inclusion

\[ C^\infty_0(\mathbb{R}^d \times (0, T)) \subset L^2_\lambda(\mathbb{R}^d \times (0, T)) \]

is dense.

**Lemma 3.29.** If \( \lambda < 0 \), then we have the inclusion \( L^2(\mathbb{R}^d \times (0, T)) \subset L^2_\lambda(\mathbb{R}^d \times (0, T)) \).

**Proof.** Let \( u \in L^2(\mathbb{R}^d \times (0, T)) \) then \( \int_{\mathbb{R}^d \times (0, T)} |u(x, t)|^2 \, dx \, dt < \infty \).

\[ 1 + |x|^2 \geq |x|^2 \geq 0 \quad \sqrt{1 + |x|^2} \geq |x| \geq 0 \]

and since \( \lambda < 0 \) we can see that \( \lambda \left( \sqrt{1 + |x|^2} \right) \leq \lambda |x| \leq 0 \) \( \exp \left( \lambda \left( \sqrt{1 + |x|^2} \right) \right) \leq \exp(\lambda |x|) \leq 1 \) In other words \( w_\lambda(x) \leq 1 \) and therefore

\[ \int_{\mathbb{R}^d \times (0, T)} |u(x, t)|^2 w_\lambda(x) \, dx \, dt \leq \int_{\mathbb{R}^d \times (0, T)} |u(x, t)|^2 \, dx \, dt < \infty \]
\[ \int_{\mathbb{R}^d \times (0,T)} |u(x,t)|^2 w_{\lambda}(x) \, dx \, dt < \infty \]
\[ u \in L^2_\lambda(\mathbb{R}^d \times (0,T)) \]

\[ L^2(\mathbb{R}^d \times (0,T)) \subset L^2_\lambda(\mathbb{R}^d \times (0,T)). \]

\[ \square \]

**Lemma 3.30.** If \( \lambda = 0 \) then \( L^2(\mathbb{R}^d \times (0,T)) = L^2_\lambda(\mathbb{R}^d \times (0,T)) \).

**Proof.** The proof is obvious because \( \lambda = 0 \) implies that \( w_{\lambda}(x) \equiv 1 \) □

**Lemma 3.31.** If \( \lambda > 0 \) we have the inclusion \( L^2_\lambda(\mathbb{R}^d \times (0,T)) \subset L^2(\mathbb{R}^d \times (0,T)) \)

**Proof.** \( 1 + |x|^2 \geq |x|^2 \geq 0 \) and since \( \lambda > 0 \) we can seen that \( \lambda \left( \sqrt{1 + |x|^2} \right) \geq \lambda |x| \geq 0 \) and \( \exp \left[ \lambda \left( \sqrt{1 + |x|^2} \right) \right] \geq \exp(\lambda |x|) \geq 1 \) In other words \( w_{\lambda}(x) \equiv 1 \).

If \( u \in L^2_\lambda(\mathbb{R}^d \times (0,T)) \) then \( \int_{\mathbb{R}^d \times (0,T)} |u(x,t)|^2 w_{\lambda}(x) \, dx \, dt < \infty \)

\[ \infty > \int_{\mathbb{R}^d \times (0,T)} |u(x,t)|^2 w_{\lambda}(x) \, dx \, dt \geq \int_{\mathbb{R}^d \times (0,T)} |u(x,t)|^2 \, dx \, dt \]

In other words

\[ \int_{\mathbb{R}^d \times (0,T)} |u(x,t)|^2 \, dx \, dt < \infty, \]

\[ u \in L^2(\mathbb{R}^d \times (0,T)), \quad L^2_\lambda(\mathbb{R}^d \times (0,T)) \subset L^2(\mathbb{R}^d \times (0,T)). \]

\[ \square \]

We will use the notation \( \| \cdot \|_{L^2(\mathbb{R}^d \times (0,T))} = \| \cdot \|_{0,2}, \) We define the Sobolev space

\[ W^{1,2}_\lambda(\mathbb{R}^d \times (0,T)) = \left\{ [u] : u, u_t, u_{x_i} \in L^2_\lambda(\mathbb{R}^d \times (0,T)) \right\} \]

endowed with the norm:

\[ \| u \|_{W^{1,2}_\lambda(\mathbb{R}^d \times (0,T))} = \left[ \| u \|_{L^2_\lambda(\mathbb{R}^d \times (0,T))}^2 + \sum_{i=1}^{d} \| u_{x_i} \|_{L^2_\lambda(\mathbb{R}^d \times (0,T))}^2 + \| u_t \|_{L^2_\lambda(\mathbb{R}^d \times (0,T))}^2 \right]^{\frac{1}{2}}. \]
And then we adapt the notation

\[ H^1_{\lambda}(\mathbb{R}^d \times (0, T)) = W^{1,2}_{\lambda}(\mathbb{R}^d \times (0, T)), \]

and for the norm we use the notation:

\[ \| \cdot \|_{W^{1,2}_{\lambda}(\mathbb{R}^d \times (0, T))} = \| \cdot \|_{1,2,\lambda}. \]

This space is a reflexive Banach space and the inclusion

\[ C^\infty_0(\mathbb{R}^d \times (0, T)) \subset H^1_{\lambda}(\mathbb{R}^d \times (0, T)) \]

is dense.

If \( \lambda < 0 \), then \( H^1(\mathbb{R}^d \times (0, T)) \subset H^1_{\lambda}(\mathbb{R}^d \times (0, T)) \)

and

If \( \lambda > 0 \), then \( H^1_{\lambda}(\mathbb{R}^d \times (0, T)) \subset H^1(\mathbb{R}^d \times (0, T)) \)

**Lemma 3.32.** If \( f \in H^1(\mathbb{R}^d \times (0, T)) \) then \( \lim_{|x| \to +\infty} f(x, t) = 0 \)

**Proof:** See [57] \( \square \)

**Lemma 3.33.** If \( u \in H^1_{\lambda}(\mathbb{R}^d \times (0, T)) \) then \( u\sqrt{w_{\lambda}} \in H^1(\mathbb{R}^d \times (0, T)) \)

**Proof:**

Note that \( u \in H^1_{\lambda}(\mathbb{R}^d \times (0, T)) \) implies that \( u, u_t, u_{x_i} \in L^2_{\lambda}(\mathbb{R}^d \times (0, T)) \) that is:

\[
\int_{\mathbb{R}^d \times (0, T)} |u(x, t)|^2 w_{\lambda}(x) \, dx \, dt < \infty
\]

\[
\int_{\mathbb{R}^d \times (0, T)} |u_t(x, t)|^2 w_{\lambda}(x) \, dx \, dt < \infty
\]

\[
\int_{\mathbb{R}^d \times (0, T)} |u_{x_i}(x, t)|^2 w_{\lambda}(x) \, dx \, dt < \infty
\]

and we want to show that
The first one is obvious. The second one is straightforward because
\[
\int_{\mathbb{R}^d \times (0,T)} \left[ u(x,t) \sqrt{w_\lambda(x)} \right]_{t}^2 \, dx \, dt < \infty
\]

So we only need to prove the third one. Let’s prove it for the case when \( d = 1 \). The general case can be proved similarly. We need to show that:
\[
\int_{\mathbb{R} \times (0,T)} \left[ u(x,t) \sqrt{w_\lambda(x)} \right]_{x}^2 \, dx \, dt < \infty
\]

\[
\left[ u(x,t) \sqrt{w_\lambda(x)} \right]_{x} = u_x(x,t) \sqrt{w_\lambda(x)} + u(x,t) \frac{w'_\lambda(x)}{2\sqrt{w_\lambda(x)}}
\]

since we want to show that
\[
\left[ u(x,t) \sqrt{w_\lambda(x)} \right]_{x} = u_x(x,t) \sqrt{w_\lambda(x)} + u(x,t) \frac{w'_\lambda(x)}{2\sqrt{w_\lambda(x)}} \in L^2(\mathbb{R} \times (0,T))
\]

It suffices to show that each of the two terms \( u_x(x,t) \sqrt{w_\lambda(x)} \) and \( u(x,t) \frac{w'_\lambda(x)}{2\sqrt{w_\lambda(x)}} \) are in \( L^2(\mathbb{R} \times (0,T)) \). The first term is obvious because we know that
\[
\int_{\mathbb{R} \times (0,T)} \left[ u_x(x,t) \sqrt{w_\lambda(x)} \right]^2 \, dx \, dt = \int_{\mathbb{R} \times (0,T)} u_x^2(x,t) \lambda_\lambda(x) \, dx \, dt < \infty.
\]

We only need to show that the second term \( u(x,t) \frac{w'_\lambda(x)}{2\sqrt{w_\lambda(x)}} \in L^2(\mathbb{R} \times (0,T)) \). We use the fact that \( w'_\lambda(x) = \frac{x_1}{\sqrt{1+x^2}} w_\lambda(x) \).

\[
\int_{\mathbb{R} \times (0,T)} \left( u(x,t) \frac{\lambda x w_\lambda(x)}{2\sqrt{(1+x^2)}w_\lambda(x)} \right)^2 \, dx \, dt = \int_{\mathbb{R} \times (0,T)} u^2(x,t) \frac{\lambda^2 x^2 w_\lambda^2(x)}{4(1+x^2)w_\lambda(x)} \, dx \, dt
\]
\[
\frac{\lambda^2}{4} \int_{\mathbb{R} \times (0,T)} u^2(x,t) \frac{x^2 w_\lambda(x)}{(1+x^2)} \, dx \, dt \leq \frac{\lambda^2}{4} \int_{\mathbb{R} \times (0,T)} u^2(x,t) w_\lambda(x) \, dx \, dt < \infty
\]

Therefore \( u \sqrt{w_\lambda} \in H^1(\mathbb{R}^d \times (0,T)) \)

**Corollary 3.34.** If \( u \in H^1_\lambda(\mathbb{R}^d \times (0,T)) \) then \( \lim_{|x| \to +\infty} u(x,t) \sqrt{w_\lambda(x)} = 0 \)

**Proof.** The proof follows from lemma 3.32.

Note that

\[
\lim_{|x| \to +\infty} \left[ u(x,t) \sqrt{w_\lambda(x)} \right] = 0,
\]

\[
\lim_{|x| \to +\infty} \left[ u_x(x,t) \sqrt{w_\lambda(x)} + u(x,t) \frac{w_\lambda'(x)}{2 \sqrt{w_\lambda(x)}} \right] = 0,
\]

\[
\lim_{|x| \to +\infty} \left[ u_x(x,t) \sqrt{w_\lambda(x)} + \frac{\lambda x w_\lambda(x)}{2 \sqrt{(1+x^2) w_\lambda(x)}} \right] = 0,
\]

\[
\lim_{|x| \to +\infty} \left[ u_x(x,t) \sqrt{w_\lambda(x)} + \frac{\lambda x \sqrt{w_\lambda(x)}}{2 \sqrt{1+x^2}} \right] = 0,
\]

\[
\lim_{|x| \to +\infty} \left[ u_x(x,t) \sqrt{w_\lambda(x)} + \frac{\lambda x}{2 \sqrt{1+x^2}} u(x,t) \sqrt{w_\lambda(x)} \right] = 0,
\]

Note that

\[
\lim_{|x| \to +\infty} \frac{\lambda x}{2 \sqrt{1+x^2}} = \lim_{|x| \to +\infty} \frac{\lambda}{2} \frac{x}{2 \sqrt{1+x^2}} = \frac{\lambda}{2} \lim_{|x| \to +\infty} \frac{x}{\sqrt{\frac{1}{x^2} + 1}}
\]

If \( x > 0 \), \( \lim_{|x| \to +\infty} \frac{x}{|x|} = \lim_{x \to +\infty} \frac{x}{x} = 1 \)

If \( x < 0 \), \( \lim_{|x| \to +\infty} \frac{x}{|x|} = \lim_{x \to -\infty} \frac{x}{-x} = -\lim_{x \to -\infty} \frac{x}{x} = -1 \)

In other words, \( \lim_{|x| \to +\infty} \frac{x}{|x|} < \infty \)

and then using the fact that \( \lim_{|x| \to +\infty} \frac{1}{\sqrt{\frac{1}{x^2} + 1}} = 1 \) we arrive at the conclusion that

\[
\frac{\lambda}{2} \lim_{|x| \to +\infty} \frac{x}{\sqrt{\frac{1}{x^2} + 1}} < \infty
\]
and then since we know that \( \lim_{|x|\to+\infty} u(x, t)\sqrt{w_{\lambda}(x)} = 0 \) we get that

\[
\lim_{|x|\to+\infty} \frac{\lambda x}{2\sqrt{1+x^2}} u(x, t)\sqrt{w_{\lambda}(x)} = 0
\]

and therefore we have gotten three important facts

1. \[
\lim_{|x|\to+\infty} u(x, t)\sqrt{w_{\lambda}(x)} = 0
\]
2. \[
\lim_{|x|\to+\infty} u_{x}(x, t)\sqrt{w_{\lambda}(x)} = 0
\]
3. \[
\lim_{|x|\to+\infty} u_{x}(x, t)\sqrt{w_{\lambda}(x)} = 0.
\]

**Lemma 3.35.** If \( \lambda < 0 \) then the imbedding \( H^{1}_{\lambda}(\mathbb{R}^{d} \times (0, T)) \subset L^{2}_{\lambda}(\mathbb{R}^{d} \times (0, T)) \) is not compact.

**Proof.** If \( \lambda < 0 \) then \( w_{\lambda}(x) \) is positive, nonincreasing and its derivative is bounded and so we know by lemma 3.27 that \( H^{1}_{\lambda}(\mathbb{R}^{d} \times (0, T)) \subset L^{2}_{\lambda}(\mathbb{R}^{d} \times (0, T)) \) is compact if and only if \( g(r) = w_{\lambda}(|x|) \) satisfies the condition that,

\[
\lim_{r \to 0} \frac{g(r+\epsilon)}{g(r)} = 0, \quad \text{for every } \epsilon > 0
\]

In this case \( g(r) = \exp \left[ \lambda \left( 1 + r^2 \right) \right] \) and \( g(r+\epsilon) = \exp \left[ \lambda \left( 1 + (r+\epsilon)^2 \right) \right] \)

\[
\frac{g(r+\epsilon)}{g(r)} = \frac{\exp \left[ \lambda \left( 1 + (r+\epsilon)^2 \right) \right]}{\exp \left[ \lambda \left( 1 + r^2 \right) \right]}
\]

\[
\lim_{r \to 0} \frac{g(r+\epsilon)}{g(r)} = \frac{\exp \left[ \lambda \left( 1 + (r+\epsilon)^2 \right) \right]}{\exp \left[ \lambda \left( 1 + r^2 \right) \right]} = \frac{\exp \left[ \lambda \left( 1 + \epsilon^2 \right) \right]}{\exp(\lambda)} \neq 0
\]

and therefore the imbedding

\[
H^{1}_{\lambda}(\mathbb{R}^{d} \times (0, T)) \subset L^{2}_{\lambda}(\mathbb{R}^{d} \times (0, T))
\]

is not compact. \( \square \)
CHAPTER 4

THE APPROXIMATE QUADRATIC SOLUTION

We start by solving the linear problem in the weighted Sobolev space. And we use the comparison theorem that we have established to define upper and lower solutions. We then use these upper and lower solutions to show that the nonlinear problem has a solution between these two functions (See [15], [16], [17], [18], [20]).

1. The Linear problem

THEOREM 4.1. The heat kernel \( \Phi(x,t) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}} \) has the following properties which can be easily verified , (See [35], [51], [33], [36], [50], [57], [59], [60], [62])

1. \( \Phi(x,t) \) is smooth for any \( x \in \mathbb{R}^d \) and \( t > 0 \)
2. \( \Phi(x,t) > 0 \) for any \( x \in \mathbb{R}^d \) and \( t > 0 \)
3. \( (\partial_t - \Delta)\Phi(x,t) = 0 \) for any \( x \in \mathbb{R}^d \) and \( t > 0 \)
4. \( \int_{\mathbb{R}^d} \Phi(x,t)dx = 1 \) for any \( t > 0 \)
5. for any \( \delta > 0 \) , \( \lim_{t \to 0^+} \int_{\mathbb{R}^d - B(0,\delta)} \Phi(x,t)dx = 0 \)
6. \( \lim_{t \to 0^+} \phi(x,t) = \delta(x) \)
7. if \( \psi \) is continous at \( 0 \) and \( \int_{\mathbb{R}^d} |\psi(x)|\Phi(x,t)dx < \infty \) then \( \lim_{t \to 0^+} \int_{\mathbb{R}^d} \psi(x)\Phi(x,t)dx = \psi(0) \).

THEOREM 4.2. The heat kernel \( \Phi \in L^2_\lambda(\mathbb{R}^d \times (0,T)) \) for any \( \lambda \in \mathbb{R} \) together with all it's derivatives.

PROOF. Lets deal with the case when \( d = 1 \). We know that \( \lambda \leq |\lambda| \) and \( \sqrt{1+x^2} \leq 1 + |x| \) putting these two together we get \( u_\lambda(x) \leq e^{\lambda(1+|x|)} \). Then we consider two cases.
case 1. 

If $t \geq \frac{1}{4\pi}$ then $4\pi t \geq \sqrt{4\pi t} \geq 1$, and therefore $\frac{1}{4\pi t} \leq \frac{1}{\sqrt{4\pi t}} \leq 1$.

\[
\int_{\mathbb{R}} \Phi^2(x,t)w_\lambda(x)dx = \frac{1}{4\pi} \int_{\mathbb{R}} e^{\frac{x^2}{4\pi}} w_\lambda(x)dx \leq \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{\frac{x^2}{4\pi} + |\lambda||x|} dx
\]

\[
= \frac{e^{|\lambda|}}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{\frac{x^2}{4\pi} + |\lambda||x|} dx = \frac{e^{|\lambda|}}{\sqrt{4\pi t}} \int_{0}^{\infty} e^{\frac{x^2}{4\pi} + |\lambda||x|} dx
\]

\[
= \frac{e^{|\lambda|} e^{\frac{x^2}{t}}}{\sqrt{4\pi t}} \int_{0}^{\infty} e^{\frac{x^2}{4\pi} + |\lambda||x|} dx
\]

and this implies,

\[
\int_{0}^{T} \int_{\mathbb{R}} \Phi^2(x,t)w_\lambda(x)dx dt \leq e^{|\lambda|} \sqrt{2} \int_{0}^{T} \frac{e^{\frac{x^2}{t}}}{\sqrt{2}t} dt < \infty.
\]

case 2. 

If $t \leq \frac{1}{4\pi}$ then $(4\pi t)^{\frac{5}{4}} \leq 4\pi t \leq \sqrt{4\pi t} \leq 1 \Rightarrow \frac{1}{4\pi t} \leq \frac{1}{\sqrt{4\pi t}} \leq 1$.

\[
\int_{\mathbb{R}} \Phi^2(x,t)w_\lambda(x)dx = \frac{1}{(4\pi t)^{\frac{5}{4}}} \int_{\mathbb{R}} e^{\frac{x^2}{4\pi}} w_\lambda(x)dx \leq \frac{1}{(4\pi t)^{\frac{5}{4}}} \int_{\mathbb{R}} e^{\frac{x^2}{4\pi} + |\lambda||x|} dx
\]

\[
= \frac{e^{|\lambda|}}{(4\pi t)^{\frac{5}{4}}} \int_{\mathbb{R}} e^{\frac{x^2}{4\pi} + |\lambda||x|} dx = \frac{2e^{|\lambda|}}{(4\pi t)^{\frac{5}{4}}} \int_{0}^{\infty} e^{\frac{x^2}{4\pi} + |\lambda||x|} dx
\]

\[
= \frac{2e^{|\lambda|}}{(4\pi t)^{\frac{5}{4}}} \int_{0}^{\infty} e^{\frac{x^2}{4\pi} + |\lambda||x|} dx
\]

\[
\int_{0}^{T} \int_{\mathbb{R}} \Phi^2(x,t)w_\lambda(x)dx dt \leq \frac{2e^{|\lambda|} e^{\frac{x^2}{t}}}{(4\pi t)^{\frac{5}{4}}} \int_{0}^{\infty} e^{\frac{x^2}{4\pi} + |\lambda||x|} dx
\]

57
Let \( y = \frac{x - |\lambda|t}{\sqrt{2t}} \), then \( dx = \sqrt{2t} dy \) and we get

\[
\int_{\mathbb{R}} \Phi^2(x, t) w_\lambda(x) dx \leq \frac{2e^{\frac{|\lambda|^2}{2} t}}{(4\pi)^{\frac{5}{4}}} \sqrt{2t} \int_{\mathbb{R}} e^{-y^2} dy \leq \frac{2\sqrt{2}}{(4\pi)^{\frac{5}{4}}} \sqrt{t} \int_{\mathbb{R}} e^{-y^2} dy
\]

\[
= \frac{2\sqrt{2}}{(4\pi)^{\frac{5}{4}}} e^{\frac{|\lambda|^2}{2} t} \sqrt{t} \sqrt{\pi}
\]

\[
= \frac{2^\frac{3}{4}}{4^{\frac{5}{4}}} \pi^{-\frac{3}{4}} e^{\frac{|\lambda|^2}{2} t} t^{-\frac{3}{4}}
\]

and this implies

\[
\int_0^T \int_{\mathbb{R}} \Phi^2(x, t) w_\lambda(x) dx dt \leq K \int_0^T t^{-3} dt = 4Kt^{\frac{1}{4}} \big|_{T=0}^{T=4KT^{\frac{1}{4}}} < \infty.
\]

A similar argument can be used to show that all its derivatives are also in \( L^2_\lambda(\mathbb{R}^d \times (0, T)) \).

**Theorem 4.3.** If \( u_0 \in L^2_\lambda(\mathbb{R}) \cap C(\mathbb{R}) \), where \( \lambda < -2c < 0 \), then the unique solution to the homogenous problem

\[
\begin{cases}
  u_t - u_{xx} = 0, & \text{in } \mathbb{R} \times (0, T), \\
  u(x, 0) = u_0(x) \\
  |u(x, t)| \leq Ke^{c|x|}, & c > 0
\end{cases}
\]

is given by \( u(x, t) = \int_{\mathbb{R}} u_0(y) \Phi(x - y, t) dy \), and \( u \in L^2_\lambda(\mathbb{R} \times (0, T)) \cap C(\mathbb{R} \times (0, T)) \) for \( \lambda < -2c < 0 \).

**Proof.** Note that \( u(x, t) = \int_{\mathbb{R}} u_0(y) \Phi(x - y, t) dy \Rightarrow \)

\[
|u(x, t)| \leq \frac{K}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t} + c|y|} dy
\]
\[
K = \frac{1}{\sqrt{4\pi t}} \left\{ \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4t} + cy} \, dy + \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4t} + cy} \, dy \right\}
\]

\[
= \frac{K}{\sqrt{4\pi t}} \left\{ \int_{-\infty}^{0} e^{-\frac{1}{4\pi t} (x^2 - 2xy + y^2 + 4cty)} \, dy + \int_{0}^{\infty} e^{-\frac{1}{4\pi t} (x^2 - 2xy + y^2 - 4cty)} \, dy \right\}
\]

\[
= \frac{K}{\sqrt{4\pi t}} \left\{ \int_{-\infty}^{0} e^{-\frac{1}{4\pi t} [(y^2 + 2(2ct-x)y + (2ct-x)^2 - (2ct-x)^2) + x^2]} \, dy \\
+ \int_{0}^{\infty} e^{-\frac{1}{4\pi t} [(y^2 - 2(2ct+x)y + (2ct+x)^2 - (2ct+x)^2) + x^2]} \, dy \right\}
\]

\[
= \frac{K}{\sqrt{4\pi t}} \left\{ \int_{-\infty}^{0} e^{-\frac{1}{4\pi t} [(y + 2ct-x)^2 - 4c^2t^2 - 4ctx + x^2]} \, dy \\
+ \int_{0}^{\infty} e^{-\frac{1}{4\pi t} [(y - (2ct+x))^2 - 4c^2t^2 + 4ctx + x^2]} \, dy \right\}
\]

\[
= \frac{K}{\sqrt{4\pi t}} \left\{ \int_{-\infty}^{0} e^{-\frac{1}{4\pi t} (y + 2ct-x)^2 + \frac{4c^2t^2}{4t} - 4ctx} \, dy \\
+ \int_{0}^{\infty} e^{-\frac{1}{4\pi t} (y - (2ct+x))^2 + \frac{4c^2t^2}{4t} + 4ctx} \, dy \right\}
\]

\[
= \frac{K}{\sqrt{4\pi t}} \left\{ e^{c^2t - cx} \int_{-\infty}^{0} e^{-\frac{1}{4\pi t} (y + 2ct-x)^2} \, dy \\
+ e^{c^2t + cx} \int_{0}^{\infty} e^{-\frac{1}{4\pi t} (y + 2ct-x)^2} \, dy \right\}
\]

\[
\leq \frac{K}{\sqrt{4\pi t}} \left\{ e^{c^2t} e^{-cx} 2\sqrt{\pi t} + e^{c^2t} e^{cx} 2\sqrt{\pi t} \right\}
\]

\[
= \tilde{K} e^{c^2t} (e^{-cx} + e^{cx}) \leq 2\tilde{K} e^{c^2t} e^{c|x|} \implies
\]
\[ \int_{\mathbb{R}} u^2(x,t)w_\lambda(x)dx \leq 4K^2 e^{2c^2t} \int_{\mathbb{R}} e^{2c|x|}w_\lambda(x)dx \leq 4K^2 e^{2c^2t} \int_{\mathbb{R}} e^{2c|x+\lambda|x|}dx = 4K^2 e^{2c^2t} \int_{\mathbb{R}} e^{(2c+\lambda)|x|}dx = M < \infty \]

provided \( 2c + \lambda < 0 \implies \lambda < -2c < 0 \) which is what we had as the range of \( \lambda \). so

\[ \int_0^T \int_{\mathbb{R}} u^2(x,t)w_\lambda(x)dx = MT < \infty. \]

Then \( u(x,t) \) is an infinitely differentiable function in the region \( t > 0 \) which is a solution to the equation, \( u \in L_\lambda^2(\mathbb{R} \times (0,T)) \) for \( \lambda < -2c < 0 \) and by number 7 of theorem 4.1 we can easily see that \( u \in C(\mathbb{R} \times (0,T)) \).

\[ \square \]

**Theorem 4.4.** Given \(|f(y,\tau)| \leq Ke^{c|y|}\) and \( u_0 \in L_\lambda^2(\mathbb{R}) \), for \( \lambda < -2c < 0 \), then the unique solution to the linear problem,

\[
\begin{cases}
    u_t - u_{xx} = f, & \text{in } \mathbb{R} \times (0,T), \\
    u(x,0) = u_0(x), \\
    |u(x,t)| \leq Ke^{c|x|}
\end{cases}
\]

is given by \( u(x,t) = \int_{\mathbb{R}} u_0(y)\Phi(x-y,t)dy + \int_0^t \int_{\mathbb{R}} f(y,\tau)\Phi(x-y,t-\tau)dyd\tau \)

**Proof.** Since we know that the first term is the solution to the homogenous problem, it suffices to show that the second term \( v(x,t) = \int_0^t \int_{\mathbb{R}} f(y,\tau)\Phi(x-y,t-\tau)dyd\tau \) is a solution to the problem.
\[
\begin{cases}
v_t - v_{xx} = f, & \text{in } \mathbb{R} \times (0, T), \\
v(x, 0) = 0, \\
|v(x, t)| \leq Ke^{c|x|}, c > 0.
\end{cases}
\]

Because the equation \( u_t - u_{xx} = 0 \) is translation invariant with respect to both \( x \) and \( t \) its easy to see the following two equations

\[
\begin{cases}
\Phi_t(x, t) - \Phi_{xx}(x, t) = 0, & \text{in } \mathbb{R} \times (0, T), \\
\Phi(x, 0) = \delta(x)
\end{cases}
\]

\[
\begin{cases}
\Phi_t(x - y, t - \tau) - \Phi_{xx}(x - y, t - \tau) = 0, & \text{in } \mathbb{R} \times (\tau, T), \\
\Phi(x - y, 0) = \delta(x - y).
\end{cases}
\]

And if we let \( U^\tau(x, t) = \int_{\mathbb{R}} f(y, \tau)\Phi(x - y, t - \tau)dy \) then by the proof of theorem 4.3 it follows that

\[
|U^\tau(x, t)| \leq M < \infty,
\]

then it follows that \( U^\tau \) is infinitely differentiable and

\[
\begin{cases}
U_{t}^\tau - U_{xx}^\tau = 0, & \text{in } \mathbb{R} \times (\tau, T), \\
U^\tau(x, \tau) = f(x, t).
\end{cases}
\]
since

\[ |U^\tau(x,t)| \leq 2\bar{K}e^{c^2 t}e^{c|x|} \]

then

\[ |v(x,t)| = \left| \int_0^t U^\tau(x,t) d\tau \right| \leq Ke^{c|x|} \]

\[ v^2(x,t) \leq \bar{K}e^{2c|x|} \]

\[ \int_{\mathbb{R}} v^2(x,t) \omega(x) dx \leq \bar{K} \int_{\mathbb{R}} e^{2c|x| + \lambda |x|} dx = \bar{K} \int_{\mathbb{R}} e^{(2c+\lambda)|x|} dx < \infty \]

provided \( 2c + \lambda < 0 \),

\( \lambda < -2c < 0 \) which was the original requirement. So \( v \in L^2_\lambda(\mathbb{R} \times (0,T)) \). Finally

\[
\begin{align*}
v_t &= f + \int_0^t \frac{\partial U^\tau}{\partial t} d\tau \quad \text{and} \\
v_{xx} &= \int_0^t \frac{\partial^2 U^\tau}{\partial x^2} d\tau
\end{align*}
\]

such that

\[
\begin{align*}
v_t - v_{xx} &= f + \int_0^t \left( \frac{\partial U^\tau}{\partial t} - \frac{\partial^2 U^\tau}{\partial x^2} \right) d\tau = f, \quad \text{and} \\
v(x,0) &= 0
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
    v_t - v_{xx} = f, & \text{in } \mathbb{R} \times (0,T), \\
v(x,0) &= 0, \\
|v(x,t)| &\leq Ke^{c|x|}, c > 0
\end{cases}
\end{align*}
\]

\[ \square \]
2. The Nonlinear problem

We use the concept of upper and lower solutions that we get from the comparison theorem that we have already derived in chapter 3. We then use the fact that we know the solution to the linear problem. We are looking to solve this problem

\[
\begin{cases}
  u_t - u_{xx} = -\hat{\rho}(x, \tau)H(\hat{g}(x, \tau) - u(x, \tau)), \quad \text{in } \mathbb{R} \times (0, \sigma^2 T^2) \\
  u(x, 0) = \hat{g}(x, 0) = g(e^x, T)e^{-ax},
\end{cases}
\]

where \( H \) is the Heaviside function given by

\[
H(x) = \begin{cases}
  1, & \text{if } x \geq 0 \\
  0, & \text{if } x < 0.
\end{cases}
\]

We are going to, if there is no chance of confusion, replace \( \tau \) with \( t \). Next we replace the Heaviside function by

\[
H_\eta(x) = \frac{1}{1 + \exp(-\frac{x}{\eta})}, \quad \text{for } \eta > 0.
\]

This is an approximation to the Heaviside because

\[
\lim_{\eta \to 0} H_\eta(x) = H(x) \quad \text{a.e.}
\]

So our equation with \( \tau \) replaced by \( t \) is

\[
\begin{cases}
  u_t - u_{xx} = -\hat{\rho}(x, t)H_\eta(\hat{g}(x, t) - u(x, t)), \quad \text{in } \mathbb{R} \times (0, \sigma^2 T^2) \\
  u(x, 0) = \hat{g}(x, 0) = g(e^x, T)e^{-ax}.
\end{cases}
\]

Note that we are going to have a severe assumption on the initial condition, We are going to assume a quadratic growth in the initial condition and we remark that this is not a good approximation.

\[
e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots
\]

\[
e^x \approx 1 + x + \frac{x^2}{2}
\]
Let’s relabel \( F(x, t, u) = -\dot{\rho}(x, t)H_\eta(\hat{g}(x, t) - u(x, t)) \) and \( u_0(x) = g(e^x, T)e^{-ax} \). Then we are looking at this kind of problem

\[
\begin{cases}
  u_t - u_{xx} = F(x, t, u), \text{in } \mathbb{R} \times (0, \frac{\sigma^2 T}{2}), \\
u(x, 0) = u_0(x) \\
|u(x, t)| \leq Ke^{c|x|}, \quad c > 0
\end{cases}
\]  

(4.1)

and in this case we can choose a number \( B > 0 \) such that \( |F| < B \) and \( |F_u| < B \).

**Lemma 4.5.** If \( |\frac{\partial}{\partial u} F(x, t, u)| < B \) then whenever we have

\[
\begin{cases}
  (u_1)_t - (u_1)_{xx} - F(x, t, u_1) \leq (u_2)_t - (u_2)_{xx} - F(x, t, u_2), \text{in } \mathbb{R} \times (0, \frac{\sigma^2 T}{2}), \\
u_1(x, 0) \leq u_2(x, 0) \\
|u_1|, |u_2| \leq Ae^{c|x|}, \quad c > 0,
\end{cases}
\]

then \( u_1 \leq u_2, \text{in } \mathbb{R} \times (0, \frac{\sigma^2 T}{2}) \)

**Proof.** Let \( w = u_1 - u_2 \) then \( w_t - w_{xx} = ((u_1)_t - (u_1)_{xx}) - ((u_2)_t - (u_2)_{xx}) \leq F(x, t, u_1) - F(x, t, u_2) = F_u(x, t, \xi)(u_1 - u_2) = F_u w \). where \( \xi = \theta u_1 + (1 - \theta)u_2 \) and \( 0 \leq \theta \leq 1 \) by mean value theorem.

So we get this equation

\[
\begin{cases}
w_t - w_{xx} - F_u w \leq 0, \text{in } \mathbb{R} \times (0, \frac{\sigma^2 T}{2}), \\
w(x, 0) \leq 0 \\
|w| \leq Ke^{c|x|}, \quad c > 0
\end{cases}
\]  

(4.2)

and we want to show that \( w \leq 0, \text{in } \mathbb{R} \times (0, \frac{\sigma^2 T}{2}). \)

We let \( z = e^{-Bt}w \). Then \( z_t = -Bz + e^{Bt}w_t, z_{xx} = e^{-Bt}w_{xx} \) so,
\[ z_t - z_{xx} = -Bz + e^{-Bt}(w_t - w_{xx}) \Rightarrow w_t - w_{xx} = e^{Bt}(Bz + z_t - z_{xx}) \]

\[ w_t - w_{xx} - F_u w = e^{Bt}((B - F_u)z + z_t - z_{xx}) \leq 0. \]

Let \( h = B - F_u > 0 \), then this is the equation we end up with,

\[
\begin{cases}
  z_t - z_{xx} + h z \leq 0, \text{ in } \mathbb{R} \times (0, \frac{a^2T}{2}), \\
  z(x, 0) \leq 0 \\
  |z| \leq Ke^{c|x|}, \\
  c > 0
\end{cases}
\]

and then by corollary 2.2, \( z \leq 0 \) in \( \mathbb{R} \times (0, \frac{a^2T}{2}) \), and therefore \( u_1 \leq u_2 \).

\[ \square \]

**Corollary 4.6.** If the function \( \tilde{u}(x, t) \) satisfies the following condition

\[
\begin{cases}
  \tilde{u}_t - \tilde{u}_{xx} - F(x, t, \tilde{u}) \geq 0, & \text{in } \mathbb{R} \times (0, \frac{a^2T}{2}), \\
  \tilde{u}(x, 0) \geq u_0(x) \\
  |\tilde{u}(x, t)| \leq K \exp(c|x|), & c > 0
\end{cases}
\]

Then \( \tilde{u}(x, t) \geq u(x, t) \) in \( \mathbb{R} \times (0, \frac{a^2T}{2}) \), and if the function \( u(x, t) \) satisfies the following condition

\[
\begin{cases}
  u_t - u_{xx} - F(x, t, u) \leq 0, & \text{in } \mathbb{R} \times (0, \frac{a^2T}{2}), \\
  u(x, 0) \leq u_0(x) \\
  |u(x, t)| \leq K \exp(c|x|), & c > 0
\end{cases}
\]

then \( u(x, t) \leq u(x, t) \) in \( \mathbb{R} \times (0, \frac{a^2T}{2}) \).
PROOF. The proof follows easily from lemma 4.5 using the fact that

\[
\begin{align*}
  u_t - u_{xx} - F(x, t, u) &= 0, \text{ in } \mathbb{R} \times (0, \frac{\sigma^2 T}{2}), \\
  u(x, 0) &= u_0(x) \\
  |u(x, t)| &\leq Ke^{c|x|}, \quad c > 0
\end{align*}
\]

The two functions \( \bar{u} \) and \( u \) are called upper and lower solutions respectively to our problem. \( \square \)

**Lemma 4.7.** If \( |F(x, t, u)| < B, \frac{\partial}{\partial u} F(x, t, u) < B \) and assume \( |u_0(x)| \leq Ax^2 \) then \( |u(x, t)| \leq Ax^2 + \tilde{B}t \) where \( \tilde{B} \geq B + 2A \)

**Proof.** The function \( \bar{u}(x, t) = Ax^2 + \tilde{B}t \) satisfies the conditions of corollary 4.6, that is,

\[
\begin{align*}
  \bar{u}_t - \bar{u}_{xx} - F(x, t, \bar{u}) &\geq 0 = u_t - u_{xx} - F(x, t, u), \text{ in } \mathbb{R} \times (0, \frac{\sigma^2 T}{2}), \\
  \bar{u}(x, 0) &= Ax^2 \geq u_0(x) = u(x, 0)
\end{align*}
\]

and the function \( \underline{u}(x, t) = -Ax^2 - \tilde{B}t \) also satisfies the condition in corollary 4.6, that is

\[
\begin{align*}
  \underline{u}_t - \underline{u}_{xx} - F(x, t, \underline{u}) &\leq 0 = u_t - u_{xx} - F(x, t, u), \text{ in } \mathbb{R} \times (0, \frac{\sigma^2 T}{2}), \\
  \underline{u}(x, 0) &= -Ax^2 \leq u_0(x) = u(x, 0)
\end{align*}
\]

Therefore \( \bar{u} \geq u \geq \underline{u} \) in \( \mathbb{R} \times (0, \frac{\sigma^2 T}{2}) \). \( \square \)

**Theorem 4.8.** If \( |F(x, t, u)| \leq B, \frac{\partial}{\partial u} F(x, t, u) \leq B \) and \( |u_0(x)| \leq Ax^2 \) then our problem has a unique solution \( u \in L^2_\lambda(\mathbb{R} \times (0, T)) \), for some \( \lambda < 0 \).

**Proof.** by lemma 4.5 we know that \( u(x, t) \leq Ax^2 + \tilde{B}t \). And so \( ||u||_{0,2,\lambda} \leq ||Ax^2 + \tilde{B}t||_{0,2,\lambda} \). Choose \( M \) such that \( M > ||Ax^2 + \tilde{B}t||_{0,2,\lambda} \) and \( T_1 = \min \left\{ \frac{5}{13B}, \frac{\sigma^2 T}{2} \right\} \) and define \( S_M = \{ v : \mathbb{R} \times (0, T_1) : ||v||_{0,2,\lambda} \leq M \} \) endowed with the parent metric.
Then we define the solution map

\[ G : S_M \rightarrow L^2_\lambda(\mathbb{R} \times (0, T_1)) \]

such that \( G(v) = u \) where \( u \) is the solution to the linear problem

\[
\begin{aligned}
    u_t - u_{xx} &= f_v, \quad \text{in } \mathbb{R} \times (0, T_1), \\
    u(x, 0) &= u_0(x) \\
    |u(x, t)| &\leq Ke^{c|x|}, \quad c > 0
\end{aligned}
\]  

and \( f_v(x, t) = F(x, t, v) \) for each fixed \( v \) since \( |f_v| \leq B \) and \( |u_0(x)| \leq Ax^2 \), corollary 4.6 guarantees that \( ||u||_{0,2,\lambda} = ||G(v)||_{0,2,\lambda} \leq ||Ax^2 + \ddot{B}t||_{0,2,\lambda} \leq M \).

So the map \( G \) is from \( S_M \) into itself. Let’s now show that it is a contraction and then we will conclude it has a unique fixed point.

\[
f_{v_1} - f_{v_2} = F(x, t, v_1) - F(x, t, v_2) = \int_{v_2}^{v_1} \frac{\partial}{\partial v} F(x, t, v) dv
\]

for \( 0 \leq \theta \leq 1 \), let \( v = \theta v_1 + (1-\theta)v_2 \) so that \( dv = (v_1 - v_2)d\theta \).

\[
f_{v_1} - f_{v_2} = (v_1 - v_2) \int_0^1 \frac{\partial}{\partial \theta} F(x, t, \theta v_1 + (1-\theta)v_2) d\theta,
\]
\[ |f_{v_1} - f_{v_2}| \leq B|v_1 - v_2|. \] Let's now look at the two solutions,

\[
\begin{cases}
(u_1)_t - (u_1)_{xx} = f_{v_1}, \text{ in } \mathbb{R} \times (0, T_1), \\
u_1(x, 0) = u_0(x) \\
|u_1(x, t)| \leq Ke^{c|x|}, & c > 0,
\end{cases}
\]

\[
\begin{cases}
(u_2)_t - (u_2)_{xx} = f_{v_2}, \text{ in } \mathbb{R} \times (0, T_1), \\
u_2(x, 0) = u_0(x) \\
|u_2(x, t)| \leq Ke^{c|x|}, & c > 0
\end{cases}
\]

Let \( w = u_1 - u_2 \) then

\[
\begin{cases}
w_t - w_{xx} = f_{v_1} - f_{v_2}, \text{ in } \mathbb{R} \times (0, T_1), \\
w(x, 0) = 0 \\
|w(x, t)| \leq Ke^{c|x|}, & c > 0
\end{cases}
\]

by lemma 4.5 again \( |w| = |u_1 - u_2| = |G(v_1) - G(v_2)| \leq BT_1|v_1 - v_2| \)

\[
||G(v_1) - G(v_2)||_{0,2,\lambda} \leq BT_1||v_1 - v_2||_{0,2,\lambda} \leq \frac{5}{13}||v_1 - v_2||_{0,2,\lambda}
\]

So \( G \) is a contraction into \( S_M \) and therefore it has a unique fixed point which is the solution to the non-linear problem in \((0, T_1)\).

\( G(u) = u \)

If \( T_1 = \frac{\sigma^2 T_2}{2} \) then we are done. If \( T_1 < \frac{\sigma^2 T}{2} \) the solution in \((0, \frac{\sigma^2 T}{2})\) is obtained by prolongation of the solution in \((0, T_1)\) as follows. \( \square \)

**Corollary 4.9.** contraction.

Since the problem is solvable in \( \mathbb{R} \times (0, T_1) \), it is also solvable in \( \mathbb{R} \times (t_1, t_2) \) where \( t_2 \leq \frac{\sigma^2 T}{2}, \ t_2 - t_1 \leq T_1 \) provided the initial condition at \( t_1 \) is less than the upper solution
and bigger than the lower solution at time \( t_1 \). In other words the problem

\[
\begin{cases}
  v_t - v_{xx} = F(x,t,v), & \text{in } \mathbb{R} \times (t_1, t_2) \\
v(x, t_1) = v_{10}(x) \\
|v(x,t)| \leq Ke^{c|x|}, & c > 0
\end{cases}
\]

where \( t_2 \leq \frac{a^2 T}{2} \), \( t_2 - t_1 \leq T_1 \) has a unique solution provided \( u(x, t_1) \leq v(x, t_1) \leq \bar{u}(x, t_1) \).

**Proof.** Let \( \bar{t} = t - t_1 \), then \( 0 \leq \bar{t} \leq T_1 \) and let \( v(x, t - t_1) = \bar{v}(x, \bar{t}) \) then \( v_t = \bar{v}_t \), \( v_{xx} = \bar{v}_{xx} \), \( F(x, t, v) = F(x, \bar{t} + t_1, \bar{v}) \) and \( v(x, t_1) = \bar{v}(x, 0) \) so we are looking at this problem

\[
\begin{cases}
  \bar{v}_t - \bar{v}_{xx} = F(x, \bar{t} + t_1, \bar{v}), & \text{in } \mathbb{R} \times (0, T_1) \\
\bar{v}(x, 0) = v_{10}(x) \\
|\bar{v}(x,t)| \leq Ke^{c|x|}, & c > 0
\end{cases}
\]

And then we have \( |F(x, \bar{t} + t_1, \bar{v})| \leq B, \frac{\partial}{\partial \bar{t}} F(x, \bar{t} + t_1, \bar{v}) | \leq B, \) and \( |\bar{v}(x, 0)| \leq Ax^2 + \bar{B} t_1 \).

By corollary 4.6 we can see that \( |\bar{v}(x, \bar{t})| \leq Ax^2 + \bar{B} t_1 + \bar{B} \bar{t} \). Choose \( M \) such that \( M > ||Ax^2 + \bar{B} t_1 + \bar{B} \bar{t}||_{0,2,\lambda} \) and define the space \( S_M \) as before

\[
S_{M} = \{w : \mathbb{R} \times (0, T_1) : ||w||_{0, 2, \lambda} \leq M\}
\]

and the solution map

\[
G : S_M \rightarrow L^2_{\lambda}(\mathbb{R} \times (0, T_1))
\]

such that \( G(w) = \bar{v} \) where \( \bar{v} \) is the solution to the linear problem

\[
\begin{cases}
  \bar{v}_t - \bar{v}_{xx} = F(x, \bar{t} + t_1, w), & \text{in } \mathbb{R} \times (0, T_1) \\
\bar{v}(x, 0) = v_{10}(x) \\
|\bar{v}(x,t)| \leq Ke^{c|x|}, & c > 0
\end{cases}
\]

69
By corollary 4.6 again we know that $|\tilde{v}(x, t)| \leq Ax^2 + \tilde{B}t_1 + \tilde{B}t$. 
$||\tilde{v}(x, t)||_{0,2,\lambda} \leq ||Ax^2 + \tilde{B}t_1 + \tilde{B}t||_{0,2,\lambda} \leq M$ and so $\tilde{v} \in S_M$. Now since $|f_{w_1} - f_{w_2}| \leq B|\tilde{v}_1 - \tilde{v}_2|$ it follows that $||G(w_1) - G(w_2)||_{0,2,\lambda} \leq BT_1||w_1 - w_2||_{0,2,\lambda} \leq \frac{5}{14}||w_1 - w_2||_{0,2,\lambda}$ so $G$ is a contraction from $S_M$ to itself and therefore it has a fixed point which is the solution to the transformed problem.

\[ \square \]

**Corollary 4.10. prolongation**

If $0 \leq T_1 \leq T_2 \leq \sigma^2 T$ and $u_1$ is the solution to the problem

\[
\begin{align*}
(u_1)_t - (u_1)_{xx} &= F(x, t, u_1), \text{ in } \mathbb{R} \times (0, T_2), \\
u_1(x, 0) &= u_0(x) \\
|u_1(x, t)| &\leq Ke^{c|x|}, \quad c > 0
\end{align*}
\]

and $u_2$ is the solution to the problem

\[
\begin{align*}
(u_2)_t - (u_2)_{xx} &= F(x, t, u_2), \text{ in } \mathbb{R} \times (T_1, \sigma^2 T), \\
u_2(x, T_1) &= u_{10}(x) \\
|u_2(x, t)| &\leq Ke^{c|x|}, \quad c > 0
\end{align*}
\]

then if $u_1(x, T_1) = u_2(x, T_1)$ then $u_1 = u_2$ in $\mathbb{R} \times (T_1, T_2)$ and the function

\[
u = \begin{cases} 
u_1, & \text{in } \mathbb{R} \times (0, T_2), \\ 
u_2, & \text{in } \mathbb{R} \times (T_2, T) \end{cases}
\]

is a solution to the problem

\[
\begin{align*}
\nu_t - \nu_{xx} &= F(x, t, \nu), \quad \text{in } \mathbb{R} \times (0, \frac{\sigma^2 T}{2}), \\
\nu(x, 0) &= u_0(x) \\
|\nu(x, t)| &\leq Ke^{c|x|}, \quad c > 0
\end{align*}
\]
Figure 4.2. The strips $\mathbb{R} \times (0, T_2)$ and $\mathbb{R} \times (T_1, \frac{\sigma^2 T}{2})$

Proof. Notice that $u_1$ and $u_2$ are both solutions to the problem

$$
\begin{align*}
\begin{cases}
(u_1)_t - (u_1)_{xx} &= F(x, t, u_1), \text{ in } \mathbb{R} \times (T_1, T_2), \\
u_1(x, T_1) &= u_{10}(x) \\
|u_1(x, t)| &\leq K e^{c|x|}, \\
&\quad c > 0
\end{cases}
\end{align*}

\begin{align*}
\begin{cases}
(u_2)_t - (u_2)_{xx} &= F(x, t, u_2), \text{ in } \mathbb{R} \times (T_1, T_2), \\
u_2(x, T_1) &= u_{10}(x) \\
|u_2(x, t)| &\leq K e^{c|x|}, \\
&\quad c > 0
\end{cases}
\end{align*}
$$
And so by uniqueness of the solution it must follow that \( u_1 = u_2 \) in \( \mathbb{R} \times (T_1, T_2) \) it's straightforward to check that this function,

\[
    u = \begin{cases} 
        u_1, & \text{in } \mathbb{R} \times (0, T_2), \\
        u_2, & \text{in } \mathbb{R} \times (T_2, \frac{\sigma^2 T}{2}) 
    \end{cases}
\]

solves this problem

\[
    \begin{cases} 
        u_t - u_{xx} = F(x, t, u), & \text{in } \mathbb{R} \times (0, \frac{\sigma^2 T}{2}), \\
        u(x, 0) = u_0(x) \\
        |u(x, t)| \leq Ke^{c|x|}, & c > 0 
    \end{cases}
\]

Now we are ready to finish our problem. we cover the strip \( \mathbb{R} \times (0, \frac{\sigma^2 T}{2}) \) with strips like this \( \mathbb{R} \times (0, T_1), \mathbb{R} \times (\frac{T_1}{2}, \frac{3T_1}{2}), \mathbb{R} \times (T_1, 2T_1), \mathbb{R} \times (\frac{3T_1}{2}, \frac{5T_1}{2}), ..., \mathbb{R} \times (\frac{mT_1}{2}, \frac{(m+2)T_1}{2}) \) where \( m = 0, 1, 2, 3, ... \) until \( \frac{m+2}{2} T_1 \geq \frac{\sigma^2 T}{2} \). Then we solve the problem in \( \mathbb{R} \times (0, T_1) \) and solve it in \( \mathbb{R} \times (\frac{T_1}{2}, \frac{3T_1}{2}) \) and then prolong to get a solution in \( \mathbb{R} \times (0, \frac{3T_1}{2}) \).
\[
\begin{cases}
(u_1)_t - (u_1)_{xx} = F(x, t, u_1), & \text{in } \mathbb{R} \times (0, T_1), \\
u_1(x, 0) = u_0(x) \\
|u_1(x, t)| \leq Ke^{c|x|}, & c > 0
\end{cases}
\]

Figure 4.4. The solution in \( \mathbb{R} \times (0, T_1) \)

\[
\begin{cases}
(u_2)_t - (u_2)_{xx} = F(x, t, u_2), & \text{in } \mathbb{R} \times (\frac{T_1}{2}, \frac{3T_1}{2}), \\
u_2(x, \frac{T_1}{2}) = u_1(x, \frac{T_1}{2}) \\
|u_2(x, t)| \leq Ke^{c|x|}, & c > 0
\end{cases}
\]
\textbf{Figure 4.5.} The solution in $\mathbb{R} \times \left( \frac{T_1}{2}, \frac{3T_1}{2} \right)$

\[
u_3 = \begin{cases} 
  u_1, & \text{in } \mathbb{R} \times (0, T_1), \\
  u_2, & \text{in } \mathbb{R} \times (T_1, \frac{3T_1}{2})
\end{cases}
\]

\textbf{Figure 4.6.} The solution in $\mathbb{R} \times \left( 0, \frac{3T_1}{2} \right)$

\[
\begin{align*}
(u_3)_t - (u_3)_{xx} &= F(x, t, u_3), & \text{in } \mathbb{R} \times (0, \frac{3T_1}{2}), \\
  u_3(x) &= u_0(x) \\
  |u_3(x, t)| &\leq Ke^{c|x|}, & c > 0
\end{align*}
\]
Then solve it in $\mathbb{R} \times (T_1, 2T_1)$ and prolong to get a solution in $\mathbb{R} \times (0, 2T_1)$.

\[
\begin{cases}
(u_4)_t - (u_4)_{xx} = F(x, t, u_4), & \text{in } \mathbb{R} \times (T_1, 2T_1), \\
u_4(x, T_1) = u_3(x, T_1) \\
|u_4(x, t)| \leq Ke^{c|x|}, & c > 0
\end{cases}
\]

\[
u_5 = \begin{cases}
u_3, & \text{in } \mathbb{R} \times (0, \frac{3T_1}{2}), \\
u_4, & \text{in } \mathbb{R} \times (\frac{3T_1}{2}, 2T_1)
\end{cases}
\]
Then solve it in \( \mathbb{R} \times \left( \frac{3T}{2}, \frac{5T}{2} \right) \) and prolong to get a solution in here \( \mathbb{R} \times (0, \frac{5T}{2}) \).

**Figure 4.9.** The solution in \( \mathbb{R} \times \left( \frac{3T}{2}, \frac{5T}{2} \right) \)

\[
\begin{cases}
(u_6)_t - (u_6)_{xx} = F(x, t, u_6), & \text{in } \mathbb{R} \times \left( \frac{3T}{2}, \frac{5T}{2} \right), \\
u_6(x, \frac{3T}{2}) = u_5(x, \frac{T}{2}) \\
|u_6(x, t)| \leq Ke^{c|x|}, & c > 0
\end{cases}
\]

\[u_7 = \begin{cases}
  u_5, & \text{in } \mathbb{R} \times (0, 2T_1), \\
  u_6, & \text{in } \mathbb{R} \times (2T_1, \frac{5T}{2})
\end{cases}\]

**Figure 4.10.** The solution in \( \mathbb{R} \times \left( 0, \frac{5T}{2} \right) \)
We then solve it in $\mathbb{R} \times (\frac{(m-1)T_1}{2}, \frac{(m+1)T_1}{2})$ and prolong it to get a solution in $\mathbb{R} \times (0, \frac{(m+1)T_1}{2})$

\begin{equation}
(u_{m-2})_t - (u_{m-2})_{xx} = F(x, t, u_{m-2}), \quad \text{in } \mathbb{R} \times (0, \frac{mT_1}{2}),
\end{equation}

\begin{equation}
u_{m-2}(x, 0) = u_0(x)\end{equation}

\begin{equation}
|u_{m-2}(x, t)| \leq Ke^{c|x|}, \quad \text{for } c > 0
\end{equation}

\textbf{Figure 4.11.} The solution in $\mathbb{R} \times \left(0, \frac{mT_1}{2}\right)$

\textbf{Figure 4.12.} The solution in $\mathbb{R} \times \left(\frac{(m-1)T_1}{2}, \frac{(m+1)T_1}{2}\right)$

77
\[
\begin{aligned}
(u_{m-1})_t - (u_{m-1})_{xx} &= F(x, t, u_{m-1}), \quad \text{in } \mathbb{R} \times \left(\frac{(m-1)T_1}{2}, \frac{(m+1)T_1}{2}\right), \\
 u_{m-1}(x, \frac{(m-1)T_1}{2}) &= u_{m-2}(x, \frac{(m-1)T_1}{2}) \\
|u_{m-1}(x, t)| &\leq K e^{c|x|}, \quad c > 0
\end{aligned}
\]

\[
\begin{aligned}
u_m = \begin{cases} 
u_{m-2}, & \text{in } \mathbb{R} \times \left(0, \frac{mT_1}{2}\right), \\ 
u_{m-1}, & \text{in } \mathbb{R} \times \left(\frac{mT_1}{2}, \frac{(m+1)T_1}{2}\right)
\end{cases}
\end{aligned}
\]

And finally since we have some $m$ such that $\frac{\sigma^2 T}{2} - \frac{m}{2} T_1 \leq T_1$ then we can solve the problem in $\mathbb{R} \times \left(\frac{mT_1}{2}, \frac{\sigma^2 T}{2}\right)$ and prolong to get a solution in $\mathbb{R} \times (0, \frac{\sigma^2 T}{2})$ and we are done.

**Figure 4.13.** The solution in $\mathbb{R} \times \left(0, \frac{(m+1)T_1}{2}\right)$
$t$

\[ u_{m+1} \]

\[ \frac{1}{2} \sigma^2 T \]

\[ \frac{1}{2} (m+1)T_1 \]

\[ \frac{1}{2} mT_1 \]

\[ x \]

\[ 0 \]

**Figure 4.14.** The solution in $\mathbb{R} \times \left( \frac{mT_1}{2}, \frac{\sigma^2 T_1}{2} \right)$

\[
\begin{cases}
(u_{m+1})_t - (u_{m+1})_{xx} = F(x, t, u_{m+1}), & \text{in } \mathbb{R} \times \left( \frac{mT_1}{2}, \frac{\sigma^2 T_1}{2} \right), \\
u_{m+1}(x, \frac{mT_1}{2}) = u_m(x, \frac{mT_1}{2}) \\
\left| u_{m+1}(x, t) \right| \leq Ke^{c|x|}, & c > 0
\end{cases}
\]

\[
u = \begin{cases}
    u_m, & \text{in } \mathbb{R} \times (0, \frac{(m+1)T_1}{2}), \\
u_{m+1}, & \text{in } \mathbb{R} \times \left( \frac{(m+1)T_1}{2}, \frac{\sigma^2 T_1}{2} \right)
\end{cases}
\]

**Figure 4.15.** The solution in $\mathbb{R} \times \left( 0, \frac{\sigma^2 T_1}{2} \right)$
3. Conclusion

We started with the Black-Scholes equation and transformed it to the nonlinear parabolic equation

\[
\begin{cases}
   (u_\eta)_t - (u_\eta)_{xx} = F_\eta(x,t,u_\eta), \text{in } \mathbb{R} \times (0, \frac{\sigma^2 T}{2}), \\
   u_\eta(x,0) = u_0(x) = g(e^x, T)e^{-ax} \\
   |u_\eta(x,t)| \leq Ae^{c|x|}, 
\end{cases}
\]

We made the strong assumption of replacing the exponential growth in the initial condition by a quadratic growth. With this assumption, we have showed that this equation has a unique approximate solution in

\[ L^2_\lambda(\mathbb{R} \times (0, \frac{\sigma^2 T}{2})) \]

for some \( \lambda < 0 \), given by

\[ u_\eta(x,t) = \int_{\mathbb{R}} u_0(y) \Phi(x-y,t)dy + \int_0^t \int_{\mathbb{R}} F(y,\tau,u_\eta) \Phi(x-y,t-\tau)dyd\tau. \]
CHAPTER 5

FUTURE DEVELOPMENTS - THE APPROXIMATE EXPONENTIAL SOLUTION

Notice that we have used a quadratic approximation to the solution. This is not a good approximation because our initial condition has an exponential growth and so it would be better if we had an exponential approximation to the solution. That is what we do in this chapter but we still use the smoothed Heaviside, $F_u + B > 0$. (see [19], [21], [22], [23], [55], [56], [47], [48], [52], [60], [53], [61], [11], [2], [13], [25])

**Theorem 5.1.** If we choose $K \geq \max\{1, \|g\|_\infty\}$, $\hat{\alpha} > |\alpha|$, $\hat{\beta} > 0$, $\hat{\beta} - \hat{\alpha}(1 + \hat{\alpha}) \geq B,$

Then the function $\bar{u}(x, t) = K \exp[\hat{\alpha} \sqrt{1 + x^2} + \hat{\beta} t]$ is an upper solution to our problem and the function $u(x, t) = -\bar{u}(x, t) = -K \exp[\hat{\alpha} \sqrt{1 + x^2} + \hat{\beta} t]$ is a lower solution to our problem.

**Proof.** Our problem is

$$
\begin{cases}
    u_t - u_{xx} - F(x, t, u) = 0, \text{ in } \mathbb{R} \times (0, \frac{\sigma^2 T}{2}), \\
    u(x, 0) = u_0(x) = g(e^x, T)e^{-ax} \\
    |u(x, t)| \leq A e^{c|x|}, \quad c > 0
\end{cases}
$$

And first we want to check if

$$
\begin{cases}
    \bar{u}_t - \bar{u}_{xx} - F(x, t, \bar{u}) \geq 0, \quad \text{in } \mathbb{R} \times (0, \frac{\sigma^2 T}{2}), \\
    \bar{u}(x, 0) \geq u_0(x) \\
    |ar{u}(x, t)| \leq A \exp(c|x|), \quad c > 0
\end{cases}
$$

81
The last two inequalities are obvious, we only check the first one, that is

\[ \bar{u}_t - \bar{u}_{xx} - F(x, t, \bar{u}) \geq 0, \quad \text{in} \quad \mathbb{R} \times (0, \frac{\sigma^2 T}{2}) \]

\( F(x, t, \bar{u}) \leq \bar{u}_t - \bar{u}_{xx} \), Note that \( \bar{u}_t = \hat{\beta} \bar{u} \) and

\[ \bar{u}_{xx} = \left[ \frac{\hat{\alpha}}{\sqrt{1 + x^2}} - \frac{\hat{\alpha} x^2}{(1 + x^2)\sqrt{1 + x^2}} + \frac{\hat{\alpha}^2 x^2}{1 + x^2} \right] \bar{u} \]

Since we want \( F(x, t, \bar{u}) \leq \bar{u}_t - \bar{u}_{xx} \), we can request \( \bar{u}_t - \bar{u}_{xx} \geq B \)

\[ \bar{u}_t - \bar{u}_{xx} = \left[ \hat{\beta} - \left( \frac{\hat{\alpha}}{\sqrt{1 + x^2}} - \frac{\hat{\alpha} x^2}{(1 + x^2)\sqrt{1 + x^2}} + \frac{\hat{\alpha}^2 x^2}{1 + x^2} \right) \right] \bar{u} \]

\[ \geq \hat{\beta} - \left( \frac{\hat{\alpha}}{\sqrt{1 + x^2}} - \frac{\hat{\alpha} x^2}{(1 + x^2)\sqrt{1 + x^2}} + \frac{\hat{\alpha}^2 x^2}{1 + x^2} \right) \]

\[ = \hat{\beta} - \frac{\hat{\alpha}}{\sqrt{1 + x^2}} \left( 1 - \frac{x^2}{1 + x^2} + \frac{\hat{\alpha} x^2}{\sqrt{1 + x^2}} \right) \]

Then use the fact that \( 1 \geq \frac{x^2}{1 + x^2} \geq 0 \implies -1 \leq -\frac{x^2}{1 + x^2} \leq 0 \implies 0 \leq 1 - \frac{x^2}{1 + x^2} \leq 1 \)

we get

\[ \bar{u}_t - \bar{u}_{xx} \geq \hat{\beta} - \frac{\hat{\alpha}}{\sqrt{1 + x^2}} \left( 1 + \frac{\hat{\alpha} x^2}{\sqrt{1 + x^2}} \right) = \hat{\beta} - \left( \frac{\hat{\alpha}}{\sqrt{1 + x^2}} + \frac{\hat{\alpha}^2 x^2}{1 + x^2} \right) \]

then since \( 1 + x^2 \geq x^2 \) and \( \sqrt{1 + x^2} \geq 1 \) we get \( \frac{x^2}{1 + x^2} \leq 1 \) and \( \frac{1}{\sqrt{1 + x^2}} \leq 1 \) then

\[ \bar{u}_t - \bar{u}_{xx} \geq \hat{\beta} - \hat{\alpha}(1 + \hat{\alpha}) \]

but we had chosen \( \hat{\alpha} \) and \( \hat{\beta} \) such that \( \hat{\beta} - \hat{\alpha}(1 + \hat{\alpha}) \geq B \) and so we conclude that \( \bar{u}_t - \bar{u}_{xx} \geq B \) and so \( F(x, t, \bar{u}) \leq \bar{u}_t - \bar{u}_{xx} \) so \( \bar{u} \) is a upper solution to our problem. Next
we want to check if \( \bar{u} \) is a lower solution to our problem by checking if

\[
\begin{cases}
  u_t - u_{xx} - F(x, t, u) \leq 0, & \text{in } \mathbb{R} \times (0, \frac{a^2T}{2}), \\
u(x, 0) \leq u_0(x) \\
|u(x, t)| \leq A \exp(c|x|), & c > 0
\end{cases}
\]

We want to show that \( F(x, t, u) \geq u_t - u_{xx} \) so we can safely request \( u_t - u_{xx} \leq -B \) but \( u_t = -\bar{u}_t, u_{xx} = -\bar{u}_{xx} \) and \( -\bar{u}_t + \bar{u}_{xx} \geq B \) \( -\bar{u}_t + \bar{u}_{xx} \leq -B \Rightarrow u_t - u_{xx} \leq -B \) and so \( u \) is a lower solution and we end up with

\[
|u(x, t)| \leq K \exp \left[ \alpha \sqrt{1 + x^2 + \beta t} \right]
\]

and for some \( \lambda < 0 \)

\[
\|u(x, t)\|_{0, 2, \lambda} \leq \|K \exp \left[ \alpha \sqrt{1 + x^2 + \beta t} \right]\|_{0, 2, \lambda} < \infty
\]

Let \( G \) be the solution map \( G(v) = u \) given by:

\[
\begin{cases}
  u_t - u_{xx} = F(x, t, v), & \text{in } \mathbb{R} \times (0, \frac{a^2T}{2}), \\
u(x, 0) = u_0(x) = g(x) e^{-\alpha x} \\
|u(x, t)| \leq A e^{c|x|}, & c > 0
\end{cases}
\]

Let \( T \) be the solution map \( T(\bar{u}) = u_1 \) given by

\[
\begin{cases}
  (u_1)_t - (u_1)_{xx} + Bu_1 = F(x, t, \bar{u}) + B\bar{u}, & \text{in } \mathbb{R} \times (0, \frac{a^2T}{2}), \\
(u_1)(x, 0) = u_0(x) \\
|u_1(x, t)| \leq A \exp(c|x|), & c > 0
\end{cases}
\]

and \( T(u_1) = u_2 \) given by

83
\[
\begin{cases}
(u_2)_t - (u_2)_{xx} + Bu_2 = F(x,t,u_1) + Bu_1, \quad \text{in} \quad \mathbb{R} \times (0, \frac{a^2 T}{2}), \\
(u_2)(x,0) = u_0(x) \\
|u_2(x,t)| \leq A \exp(c|x|), \quad c > 0
\end{cases}
\]

This leads to a sequence \( \{u_n\}_{n=1}^{\infty} \). And we similarly define another sequence by

\[
T(u) = v_1 \quad \text{given by}
\]

\[
\begin{cases}
(v_1)_t - (v_1)_{xx} + Bv_1 = F(x,t,u) + Bu, \quad \text{in} \quad \mathbb{R} \times (0, \frac{a^2 T}{2}), \\
(v_1)(x,0) = u_0(x) \\
|v_1(x,t)| \leq A \exp(c|x|), \quad c > 0
\end{cases}
\]

and \( T(v_1) = v_2 \) given by

\[
\begin{cases}
(v_2)_t - (v_2)_{xx} + Bv_2 = F(x,t,v_1) + Bv_1, \quad \text{in} \quad \mathbb{R} \times (0, \frac{a^2 T}{2}), \\
(v_2)(x,0) = u_0(x) \\
|v_2(x,t)| \leq A \exp(c|x|), \quad c > 0
\end{cases}
\]

and this leads to another sequence \( \{v_n\}_{n=1}^{\infty} \).

**Theorem 5.2.**

1. The sequence \( \{u_n\}_{n=1}^{\infty} \) is a monotone decreasing sequence bounded below by \( \bar{u} \) and therefore it converges to some function \( u^* \).
2. The sequence \( \{v_n\}_{n=1}^{\infty} \) is a monotone increasing sequence bounded above by \( \bar{u} \) and therefore it converges to some function \( v^* \).
3. \( u^* = v^* = u \) which is the unique solution to the nonlinear problem in

\[ H^1_\lambda(\mathbb{R} \times (0, \frac{a^2 T}{2})). \]
PROOF. Let’s first show \( u_1 \leq \bar{u} \) by showing \( \bar{u} - u_1 \geq 0 \)

\[
\begin{align*}
\bar{u}_t - \bar{u}_{xx} + B \bar{u} &\geq F(x,t,\bar{u}) + B \bar{u}, \quad \text{in} \quad \mathbb{R} \times (0, \frac{\sigma^2 T}{2}), \\
\bar{u}(x,0) &\geq u_0(x) \\
|\bar{u}(x,t)| &\leq A \exp(c|x|), \quad c > 0
\end{align*}
\]

Let \( w = \bar{u} - u_1 \) then we get

\[
\begin{align*}
\bar{u}_t - \bar{u}_{xx} + B \bar{u} &\geq F(x,t,\bar{u}) + B \bar{u}, \quad \text{in} \quad \mathbb{R} \times (0, \frac{\sigma^2 T}{2}), \\
\bar{u}(x,0) - u_1(x,0) &\geq 0 \\
|\bar{u}(x,t) - u_1(x,t)| &\leq A \exp(c|x|), \quad c > 0
\end{align*}
\]

then by corollary 2.2 we get \( w \geq 0, \quad \bar{u} - u_1 \geq 0, \quad u_1 \leq \bar{u} \)

Next we want to show that \( u_1 \geq y \) by showing \( v_1 - u_1 \geq 0 \)

\[
\begin{align*}
(v_1)_t - (v_1)_{xx} + B v_1 &= F(x,t,u) + B u, \quad \text{in} \quad \mathbb{R} \times (0, \frac{\sigma^2 T}{2}), \\
(v_1)(x,0) &= u_0(x) \\
|v_1(x,t)| &\leq A \exp(c|x|), \quad c > 0
\end{align*}
\]
\[
\begin{aligned}
&\begin{cases}
  u_t - u_{xx} + Bu \leq F(x,t,u) + Bu, & \text{in } \mathbb{R} \times (0, \frac{a^2 T}{2}), \\
  u(x,0) \leq u_0(x) \\
  |u(x,t)| \leq A \exp(c|\omega|), & c > 0
\end{cases} \\
\end{aligned}
\]

subtracting these two equations we get

\[
\begin{aligned}
&\begin{cases}
  (v_1 - u)_t - (v_1 - u)_{xx} + B(v_1 - u) \geq 0, & \text{in } \mathbb{R} \times (0, \frac{a^2 T}{2}), \\
  (v_1 - u)(x,0) \geq 0 \\
  |v_1(x,t) - u(x,t)| \leq A \exp(c|\omega|), & c > 0
\end{cases} \\
\end{aligned}
\]

Let \( w = v_1 - u \) then

\[
\begin{aligned}
&\begin{cases}
  w_t - w_{xx} + Bw \geq 0, & \text{in } \mathbb{R} \times (0, \frac{a^2 T}{2}), \\
  w(x,0) \geq 0 \\
  |w(x,t)| \leq A \exp(c|\omega|), & c > 0
\end{cases} \\
\end{aligned}
\]

then by corollary 2.2 \( w \geq 0 \quad v_1 - u \geq 0 \quad v_1 \geq u \)

Next we prove that if \( u \leq v \) then \( Tu \leq Tv \) that is \( T \) is strictly monotone.

\[
\begin{aligned}
&\begin{cases}
  (Tu)_t - (Tu)_{xx} + BTu = F(x,t,u) + Bu, & \text{in } \mathbb{R} \times (0, \frac{a^2 T}{2}), \\
  Tu(x,0) = u_0(x) \\
  |Tu(x,t)| \leq A \exp(c|\omega|), & c > 0
\end{cases} \\
\end{aligned}
\]

\[
\begin{aligned}
&\begin{cases}
  (Tv)_t - (Tv)_{xx} + BTv = F(x,t,v) + Bv, & \text{in } \mathbb{R} \times (0, \frac{a^2 T}{2}), \\
  Tv(x,0) = u_0(x) \\
  |Tv(x,t)| \leq A \exp(c|\omega|), & c > 0
\end{cases} \\
\end{aligned}
\]

86
subtracting these two equations we get

\[
\begin{cases}
(Tv - Tu)_t - (Tv - Tu)_{xx} + B(Tv - Tu) = (F(x, t, v) + Bu) - (F(x, t, u) + Bu), \\
Tv(x, 0) - Tu(x, 0) = 0 \\
|Tv(x,t) - Tu(x,t)| \leq A \exp(c|x|), \quad c > 0
\end{cases}
\]

let \( w = Tv - Tu \) and \( F^*(x, t, u) = F(x, t, u) + Bu \) then \( F_u^* = F_u + B > 0 \) so \( F^* \) is increasing and therefore \((F(x, t, v) + Bu) - (F(x, t, u) + Bu) \geq 0\)

\[
\begin{cases}
w_t - w_{xx} + Bw \geq 0, \quad \text{in} \quad \mathbb{R} \times (0, \frac{\sigma^2 T}{2}), \\
w(x, 0) = 0 \\
|w(x,t)| \leq A \exp(c|x|), \quad c > 0
\end{cases}
\]

So by corollary 2.2 again, \( w > 0 \) unless \( w \equiv 0 \) which means \( Tu = Tv \) which in turn implies that \( F^*(x, t, u) = F^*(x, t, v) \) which occurs only if \( u \equiv v \) therefore \( Tv > Tu \) and \( T \) is strictly monotone.

The last claim we want to prove is that \( v_n \leq u_n \) for all \( n \). We use induction to prove this. We know that \( v_0 \leq u_0 \) we assume \( v_{n-1} \leq u_{n-1} \) and we want to show that \( v_n \leq u_n \). Using the monotonicity of the solution map \( T \) we get \( v_n = Tv_{n-1} \leq Tu_{n-1} = u_n \) and we are done.

Now we have gotten that \( \{u_n\}_{n=1}^{\infty} \) is a decreasing sequence bounded below pointwise by \( v_0(x) \) and \( \{v_n\}_{n=1}^{\infty} \) is an increasing sequence bounded above pointwise by \( u_0(x) \) so \( u_n \) converges pointwise to a function \( u^*(x) \) and \( v_n \) converges pointwise to a function \( v^*(x) \).

**Claim**

\[
Tu^* = u^* \quad \text{and} \quad Tv^* = v^* \quad \text{so} \quad u^* \quad \text{and} \quad v^* \quad \text{are fixed points of} \quad T.
\]

Based on this claim, which we still have to prove, we can establish the results of our
theorem. The claim gives us the following:

\[
\begin{align*}
&\begin{cases}
(u^*)_t - (u^*)_{xx} + Bu^* = F(x, t, u^*) + Bu^*, & \text{in } \mathbb{R} \times (0, \sigma^2 T/2), \\
(u^*)(x, 0) = u_0(x) \\
|u^*(x, t)| \leq A \exp(c|x|), & c > 0
\end{cases}
\end{align*}
\]

\[
\begin{align*}
&\begin{cases}
(v^*)_t - (v^*)_{xx} + Bv^* = F(x, t, v^*) + Bv^*, & \text{in } \mathbb{R} \times (0, \sigma^2 T/2), \\
(v^*)(x, 0) = u_0(x) \\
|v^*(x, t)| \leq A \exp(c|x|), & c > 0
\end{cases}
\end{align*}
\]

We are going to depend on the fact that any fixed point of \( T \) is a fixed point of \( G \), that is \( Tu = u \) if and only if \( Gu = u \). so

\[
Gu^* = u^* \quad \text{and} \quad Gv^* = v^*
\]

\[
\begin{align*}
&\begin{cases}
(u^*)_t - (u^*)_{xx} = F(x, t, u^*), & \text{in } \mathbb{R} \times (0, \sigma^2 T/2), \\
(u^*)(x, 0) = u_0(x) \\
|u^*(x, t)| \leq A \exp(c|x|), & c > 0
\end{cases}
\end{align*}
\]

\[
\begin{align*}
&\begin{cases}
(v^*)_t - (v^*)_{xx} = F(x, t, v^*), & \text{in } \mathbb{R} \times (0, \sigma^2 T/2), \\
(v^*)(x, 0) = u_0(x) \\
|v^*(x, t)| \leq A \exp(c|x|), & c > 0
\end{cases}
\end{align*}
\]

We know that

\[
\begin{align*}
&\begin{cases}
(u^*)_t - (u^*)_{xx} - F(x, t, u^*) \geq 0, & \text{in } \mathbb{R} \times (0, \sigma^2 T/2), \\
(u^*)(x, 0) \geq u_0(x) \\
|u^*(x, t)| \leq A \exp(c|x|), & c > 0
\end{cases}
\end{align*}
\]
and
\[
\begin{align*}
(v^*)_t - (v^*)_{xx} - F(x, t, v^*) & \leq 0, \quad \text{in } \mathbb{R} \times (0, \frac{a^2 T}{2}), \\
(v^*)(x, 0) & \leq u_0(x) \\
|v^*(x, t)| & \leq A \exp(c|x|), \quad c > 0
\end{align*}
\]
by lemma 4.5. These two equations imply
\[ u^* \geq v^*. \] (5.1)

We also have,
\[
\begin{align*}
(u^*)_t - (u^*)_{xx} - F(x, t, u^*) & \leq 0, \quad \text{in } \mathbb{R} \times (0, \frac{a^2 T}{2}), \\
(u^*)(x, 0) & \leq u_0(x) \\
|u^*(x, t)| & \leq A \exp(c|x|), \quad c > 0
\end{align*}
\]
and
\[
\begin{align*}
(v^*)_t - (v^*)_{xx} - F(x, t, v^*) & \geq 0, \quad \text{in } \mathbb{R} \times (0, \frac{a^2 T}{2}), \\
(v^*)(x, 0) & \geq u_0(x) \\
|v^*(x, t)| & \leq A \exp(c|x|), \quad c > 0
\end{align*}
\]
by lemma 4.5. These two equations imply
\[ u^* \leq v^*. \] (5.2)

From 5.1 and 5.2 we conclude that \( u^* = v^* = u \) which would be the unique solution to our problem.
\[
\begin{align*}
& u_t - u_{xx} = F(x, t, u), \quad \text{in } \mathbb{R} \times (0, \frac{a^2 T}{2}), \\
& u(x, 0) = u_0(x) \\
& |u(x, t)| \leq A \exp(c|x|), \quad c > 0.
\end{align*}
\]
References


91


